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Relativistic Corrections to the Sunyaev-Zeldovich Effect for Clusters of Galaxies

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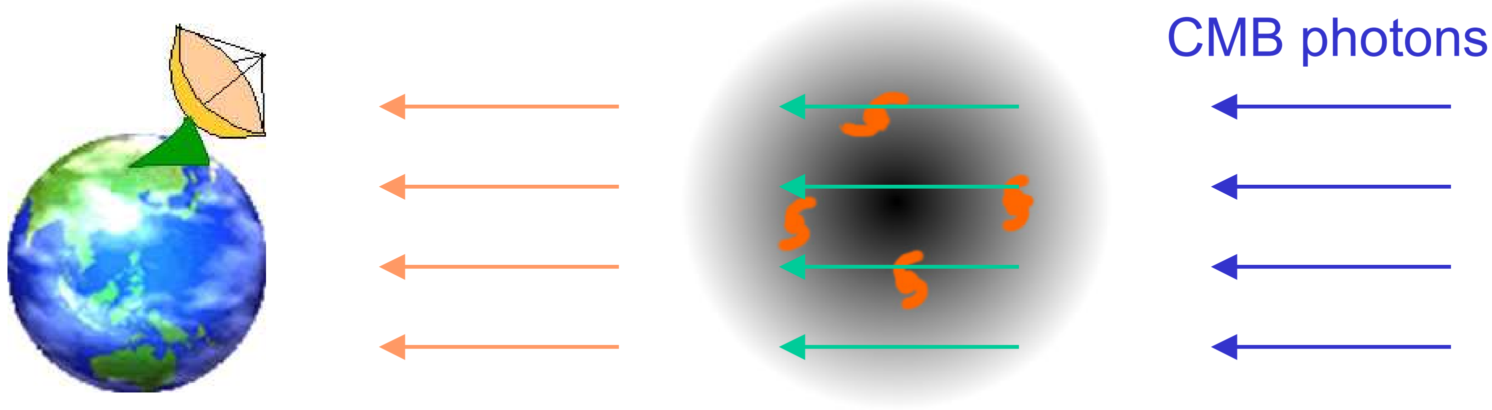
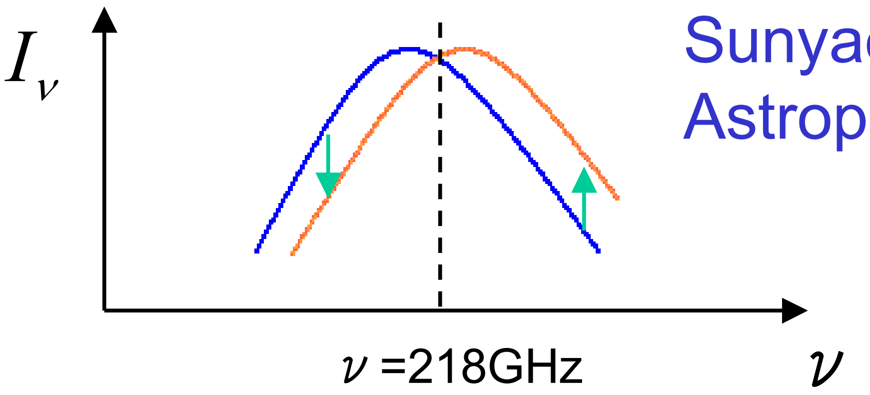
Contents

(1) Introduction

Distortion of the Cosmic Microwave Background (CMB) photon spectrum due to electrons in the CG.
(Inverse Compton scattering)





Sunyaev & Zeldovich, Comments Astrophys. Space Phys. 4, 173 (1972)

ΔT_{CMB} is determined

Determination of the Hubble constant with CG

$$\begin{cases} \Delta T_{CMB} \propto N_e \; l \; T_e & \text{--- Thermal SZ effect} \\ S_{X-ray} \propto N_e^2 \; l \; T_e^{1/2} & \text{--- X-ray intensity} \end{cases}$$

$$\therefore \quad l \propto \left(\Delta T_{CMB}\right)^2 \; S_{X-ray} \; T_e^{-3/2}$$

l : diameter of CG along the line of sight

$$D_A = \frac{l}{\theta} \quad \text{(angular diameter distance)}$$

With the redshift observation, the Hubble constant is determined (H_0=v/D_A).

Recent observations:

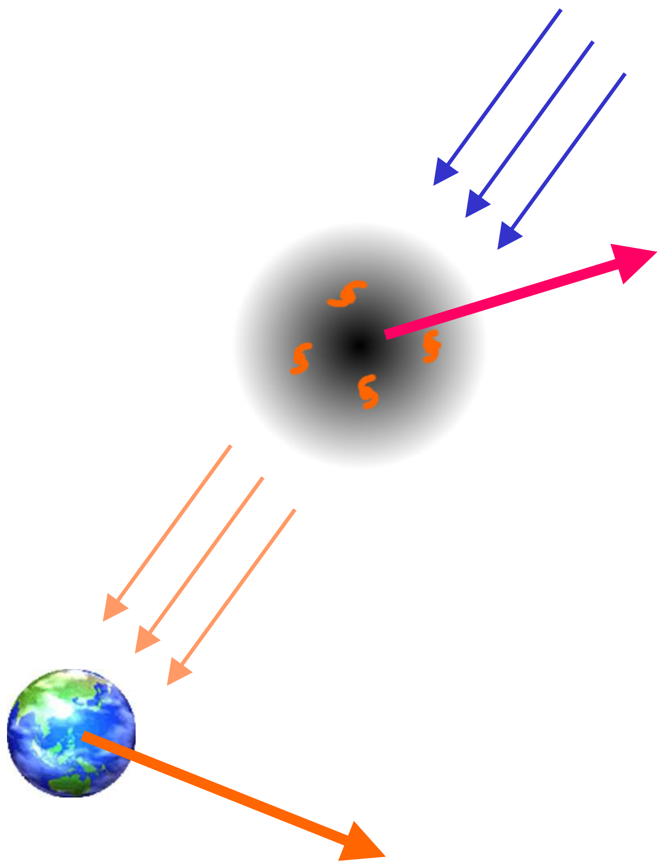
High temperature CG T_e=10～20keV.

Relativisitic corrections become significant!

Typical order of magnitude

$$\theta_e \equiv T_e / m_e \quad = \quad 10/511 \approx 0.02$$

(2) Covariant Formalism for SZ Effects



① Boltzman equation (CMB frame)
→ Thermal SZ

② Lorenz boosting CG
peculiar motion of CG
→ Kinematical SZ (CG)

③ Lorenz boosting the observer
motion of observer
→ Kinematical SZ (observer)

④ polarization cross sections &
Stokes parameters for CMB
→ Polarization SZ

⑤ Iteration of Boltzman equation
→ Double scattering SZ

(a)Thermal SZ Effect

Itoh, Kohyama & Nozawa, ApJ 502, 7 (1998)

Boltzman equation for the photon distribution function.

$$\frac{\partial n(\omega)}{\partial t} = -2\int \frac{d^3p}{(2\pi)^3} d^3p' d^3k' W \{n(\omega)[1+n(\omega')]f(E) - n(\omega')[1+n(\omega)]f(E')\},$$

$$W = \frac{(e^2/4\pi)^2}{2\omega\omega' E E'} \overline{R}\ \delta^4(p+k-p'-k')\ ,$$

$\overline{R}$: Inverse Compton scattering amplitude

$f(E),\ f(E')$: Fermi distribution functions

$p,\ p'$: electron momenta, $k,\ k'$: photon momenta

Fokker-Planck expansion

$(\Delta x << 1)$

$$\Delta x \equiv \frac{\omega' - \omega}{k_B T_e} \ , \ \ x \equiv \frac{\omega}{k_B T_e}$$

$$\begin{aligned}
\frac{\partial n(\omega)}{\partial t} &= 2\left[\frac{\partial n}{\partial x} + n(1+n)\right] I_1 \\
&+ 2\left[\frac{\partial^2 n}{\partial x^2} + 2(1+n)\frac{\partial n}{\partial x} + n(1+n)\right] I_2 \\
&+ 2\left[\frac{\partial^3 n}{\partial x^3} + 3(1+n)\frac{\partial^2 n}{\partial x^2} + 3(1+n)\frac{\partial n}{\partial x} + n(1+n)\right] I_3 \\
&+ 2\left[\frac{\partial^4 n}{\partial x^4} + 4(1+n)\frac{\partial^3 n}{\partial x^3} + 6(1+n)\frac{\partial^2 n}{\partial x^2} + 4(1+n)\frac{\partial n}{\partial x} + n(1+n)\right] I_4 \\
&+ \cdots ,
\end{aligned}$$

$$I_k \equiv \frac{1}{k!}\int \frac{d^3p}{(2\pi)^3} d^3p' d^3k' \, W \, f(E) \, (\Delta x)^k .$$

Distortion of the CMB spectrum

Expansion in terms of $\theta_e \equiv \dfrac{k_B T_e}{m_e c^2}$

$$\frac{\Delta n(X)}{n_0(X)} = \frac{\tau \theta_e X e^X}{e^X - 1}[Y_0 + \underline{\theta_e Y_1 + \theta_e^2 Y_2 + \theta_e^3 Y_3 + \theta_e^4 Y_4}] ,$$

Relativistic corrections

$$X = \frac{\hbar\omega}{k_B T_0} , \quad \tau \equiv \sigma_T \int dlNe , \quad \sigma_T : \text{Thomson crosssection}$$

T_0 : CMB temperature

$$Y_0 = -4 + \tilde{X} ,$$

--- Non-relativistic term

$$\tilde{X} \equiv X \coth\left(\frac{X}{2}\right)$$

Relativistic corrections

$$
\begin{aligned}
Y_1 &= -10 + \frac{47}{2}\tilde{X} - \frac{42}{5}\tilde{X}^2 + \frac{7}{10}\tilde{X}^3 + \tilde{S}^2\left(-\frac{21}{5} + \frac{7}{5}\tilde{X}\right) , \\
Y_2 &= -\frac{15}{2} + \frac{1023}{8}\tilde{X} - \frac{868}{5}\tilde{X}^2 + \frac{329}{5}\tilde{X}^3 - \frac{44}{5}\tilde{X}^4 + \frac{11}{30}\tilde{X}^5 \\
&+ \tilde{S}^2\left(-\frac{434}{5} + \frac{658}{5}\tilde{X} - \frac{242}{5}\tilde{X}^2 + \frac{143}{30}\tilde{X}^3\right) \\
&+ \tilde{S}^4\left(-\frac{44}{5} + \frac{187}{60}\tilde{X}\right) ,
\end{aligned}
$$

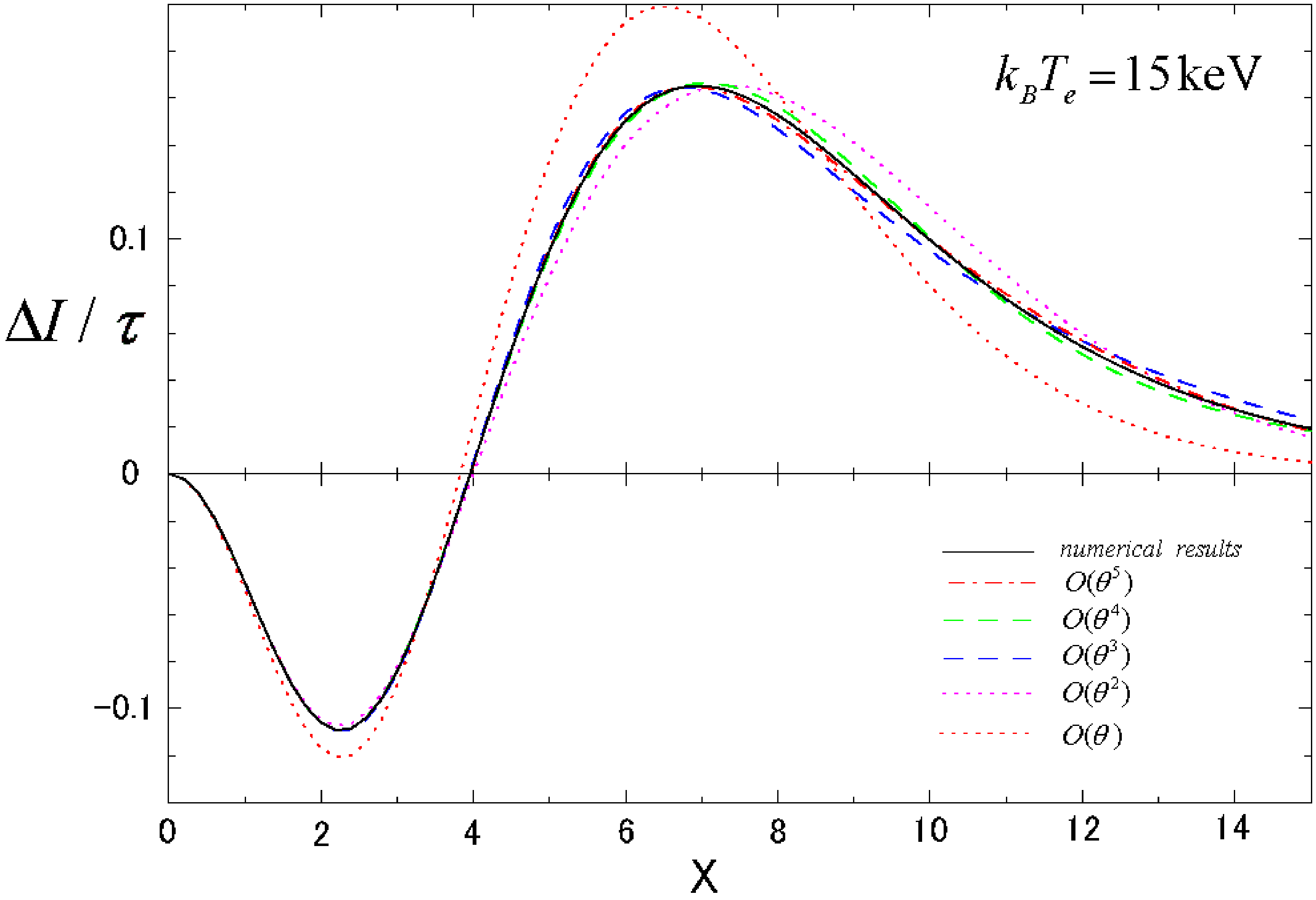
$$
\begin{aligned}
Y_3 &= \frac{15}{2} + \frac{2505}{8}\tilde{X} - \frac{7098}{5}\tilde{X}^2 + \frac{14253}{10}\tilde{X}^3 - \frac{18594}{35}\tilde{X}^4 \\
&+ \frac{12059}{140}\tilde{X}^5 - \frac{128}{21}\tilde{X}^6 + \frac{16}{105}\tilde{X}^7 \\
&+ \tilde{S}^2\left(-\frac{7098}{10} + \frac{14253}{5}\tilde{X} - \frac{102267}{35}\tilde{X}^2 + \frac{156767}{140}\tilde{X}^3 - \frac{1216}{7}\tilde{X}^4 + \frac{64}{7}\tilde{X}^5\right) \\
&+ \tilde{S}^4\left(-\frac{18594}{35} + \frac{205003}{280}\tilde{X} - \frac{1920}{7}\tilde{X}^2 + \frac{1024}{35}\tilde{X}^3\right) \\
&+ \tilde{S}^6\left(-\frac{544}{21} + \frac{992}{105}\tilde{X}\right) ,
\end{aligned}
$$

$$
\begin{aligned}
Y_4 \;=\; & -\frac{135}{32}+\frac{30375}{128}\tilde{X}-\frac{62391}{10}\tilde{X}^2+\frac{614727}{40}\tilde{X}^3-\frac{124389}{10}\tilde{X}^4 \\
+\; & \frac{355703}{80}\tilde{X}^5-\frac{16568}{21}\tilde{X}^6+\frac{7516}{105}\tilde{X}^7-\frac{22}{7}\tilde{X}^8+\frac{11}{210}\tilde{X}^9 \\
+\; & \tilde{S}^2\left(-\frac{62391}{20}+\frac{614727}{20}\tilde{X}-\frac{1368279}{20}\tilde{X}^2+\frac{4624139}{80}\tilde{X}^3-\frac{157396}{7}\tilde{X}^4\right. \\
& \quad \left.+\;\frac{30064}{7}\tilde{X}^5-\frac{2717}{7}\tilde{X}^6+\frac{2761}{210}\tilde{X}^7\right) \\
+\; & \tilde{S}^4\left(-\frac{124389}{10}+\frac{6046951}{160}\tilde{X}-\frac{248520}{7}\tilde{X}^2+\frac{481024}{35}\tilde{X}^3-\frac{15972}{7}\tilde{X}^4\right. \\
& \quad \left.+\;\frac{18689}{140}\tilde{X}^5\right) \\
+\; & \tilde{S}^6\left(-\frac{70414}{21}+\frac{465992}{105}\tilde{X}-\frac{11792}{7}\tilde{X}^2+\frac{19778}{105}\tilde{X}^3\right) \\
+\; & \tilde{S}^8\left(-\frac{682}{7}+\frac{7601}{210}\tilde{X}\right)\;,
\end{aligned}
$$

$$
\tilde{S} \;\equiv\; \frac{X}{\sinh\left(\dfrac{X}{2}\right)}
$$

Distortion of the spectral intensity

$$\Delta I \equiv \frac{X^3}{e^X - 1} \frac{\Delta n(X)}{n_0(X)}$$



Summary for thermal SZ effect

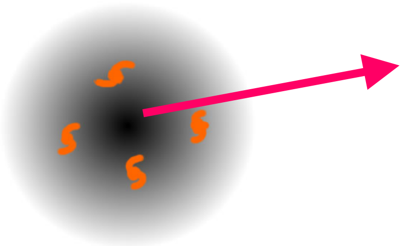
- Relativistic corrections are very important for high-temperature CG T_e> 10keV.
- Relativistic corrections are significant for large X region (sub-millimeter region)
  For X>10, factor 4 effect!
- Fokker-Planck expansion approximation is OK for T_e < 15keV.

(b) Kinematical SZ Effect
Peculiar Motion of Clusters of Galaxies

Nozawa, Itoh & Kohyama ApJ, 508, 17 (1998)

Lorentz boosting of the original Boltzman equation

$$\begin{cases} f(E) = f_{CG}(E_{CG}) \ , \ f(E') = f_{CG}(E'_{CG}) \\ E_{CG} = \dfrac{E - \vec{\beta} \cdot \vec{p}}{\sqrt{1-\beta^2}} \ , \ E'_{CG} = \dfrac{E' - \vec{\beta} \cdot \vec{p}\,'}{\sqrt{1-\beta^2}} \ , \quad \vec{\beta} = \dfrac{\vec{v}}{c} \end{cases}$$



$\vec{v}$ Peculiar velocity of CG relative to CMB.

Distortion of the CMB spectrum

$$\frac{\Delta n(X)}{n_0(X)} = \frac{\boldsymbol{\tau} X e^X}{e^X - 1}\theta_e \left[Y_0 + \theta_e Y_1 + \theta_e^2 Y_2 + \theta_e^3 Y_3 + \theta_e^4 Y_4 \right]$$
$$+ \frac{\boldsymbol{\tau} X e^X}{e^X - 1}\beta^2 \left[\frac{1}{3} Y_0 + \theta_e \left(\frac{5}{6} Y_0 + \frac{2}{3} Y_1 \right) \right]$$
$$+ \frac{\boldsymbol{\tau} X e^X}{e^X - 1}\beta P_1(\hat{\beta}_z) \left[1 + \boxed{\theta_e C_1 + \theta_e^2 C_2]} \right.$$
$$+ \frac{\boldsymbol{\tau} X e^X}{e^X - 1}\beta^2 P_2(\hat{\beta}_z) \boxed{\left[D_0 + \theta_e D_1 \right] ,}$$

$$\hat{\beta}_z \equiv \frac{\beta_z}{\beta} = \cos\theta_\gamma ,$$

$$P_1(\hat{\beta}_z) = \hat{\beta}_z ,$$

$$P_2(\hat{\beta}_z) = \frac{1}{2}\left(3\hat{\beta}_z^2 - 1\right) ,$$

Kinematical SZ effect

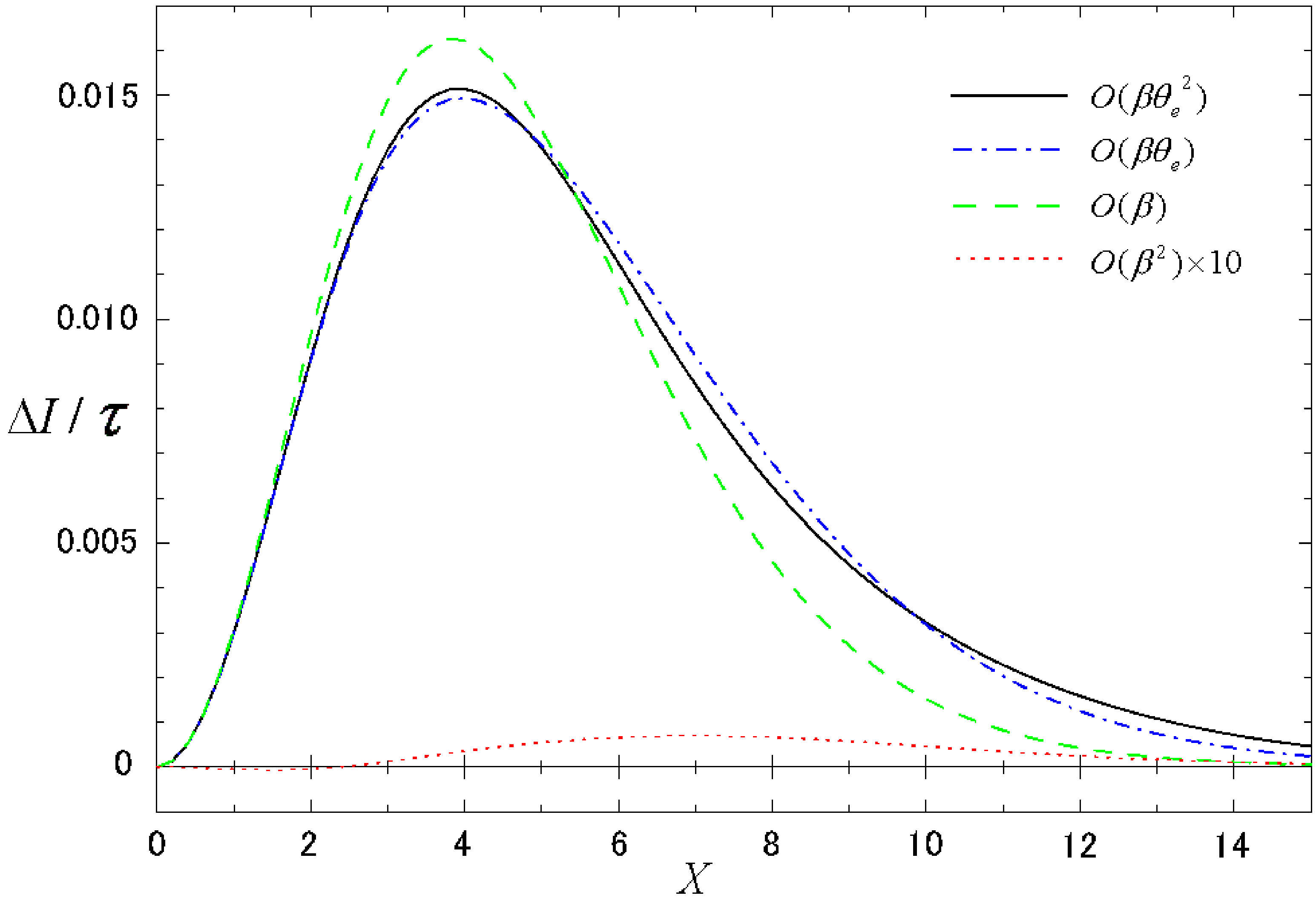
Relativistic corrections

$$
\begin{aligned}
C_1 &= 10 - \frac{47}{5}\tilde{X} + \frac{7}{5}\tilde{X}^2 + \frac{7}{10}\tilde{S}^2\,, \\
C_2 &= 25 - \frac{1117}{10}\tilde{X} + \frac{847}{10}\tilde{X}^2 - \frac{183}{10}\tilde{X}^3 + \frac{11}{10}\tilde{X}^4 \\
&\quad + \tilde{S}^2\left(\frac{847}{20} - \frac{183}{5}\tilde{X} + \frac{121}{20}\tilde{X}^2\right) + \frac{11}{10}\tilde{S}^4\,, \\
D_0 &= -\frac{2}{3} + \frac{11}{30}\tilde{X}\,, \\
D_1 &= -4 + 12\tilde{X} - 6\tilde{X}^2 + \frac{19}{30}\tilde{X}^3 + \tilde{S}^2\left(-3 + \frac{19}{15}\tilde{X}\right)\,.
\end{aligned}
$$

Distortion of the spectral intensity

$$\Delta I \equiv \frac{X^3}{e^X - 1}\frac{\Delta n(X)}{n_0(X)}$$

$k_B T_e = 10$ keV, v = 1,000 km/s



Summary for kinematical SZ effect (CG)

- Relativistic corrections are very important for high-temperature CG T_e> 10keV.

$$\text{For } k_B T_e = 10\text{keV}, \ v = 1{,}000\text{km/s}$$

$$O(\beta\theta_e) = -8.2\%$$

$$O(\beta\theta_e^2) = +1.3\%$$

Motion of the observer (Solar System)

Nozawa, Itoh & Kohyama submitted to A&A (2005)

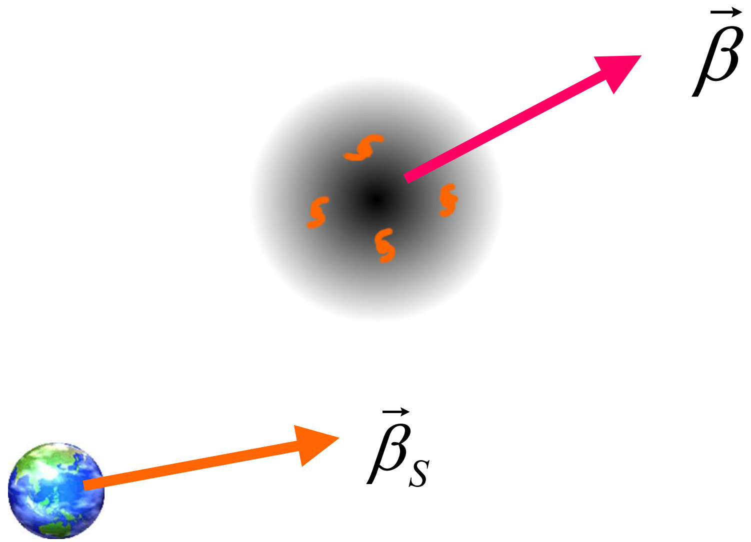
$\vec{\beta}_S$: velocity of the solar system respect to CMB

$\vec{k}_S$: photon vector in the solar system

$$n(\omega) = n(\omega_S)\,,$$

$$\omega = \gamma_S(\omega_S + \vec{\beta}_S \cdot \vec{k}_S)\,,$$

$$\gamma_S \equiv \frac{1}{\sqrt{1-\beta_S{}^2}}\,,$$



Planck distribution: dipolar distortion

$$n_S{}^0(X_S,\mu_S)=\frac{1}{\exp\{\gamma_S X_S(1+\beta_S\mu_S)\}-1},$$

$$X_S\equiv\frac{\omega_S}{k_B T_0},$$

$$\mu_S\equiv\hat{\beta}_S\cdot\hat{k}_S=\cos\theta_S\ ,$$

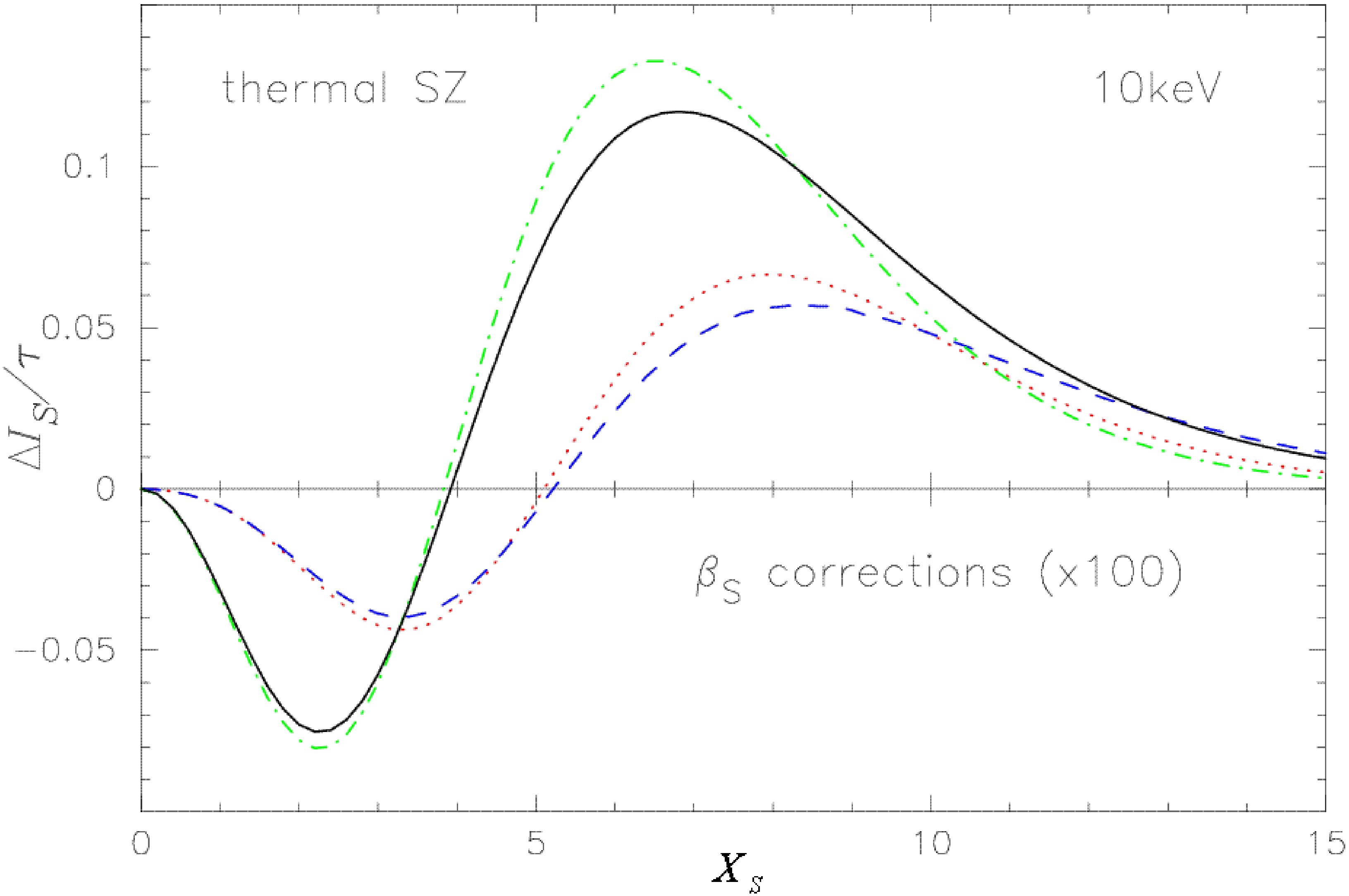
$$X=\gamma_S X_S(1+\beta_S\mu_S)$$

velocity of CG in CMB frame is also transformed

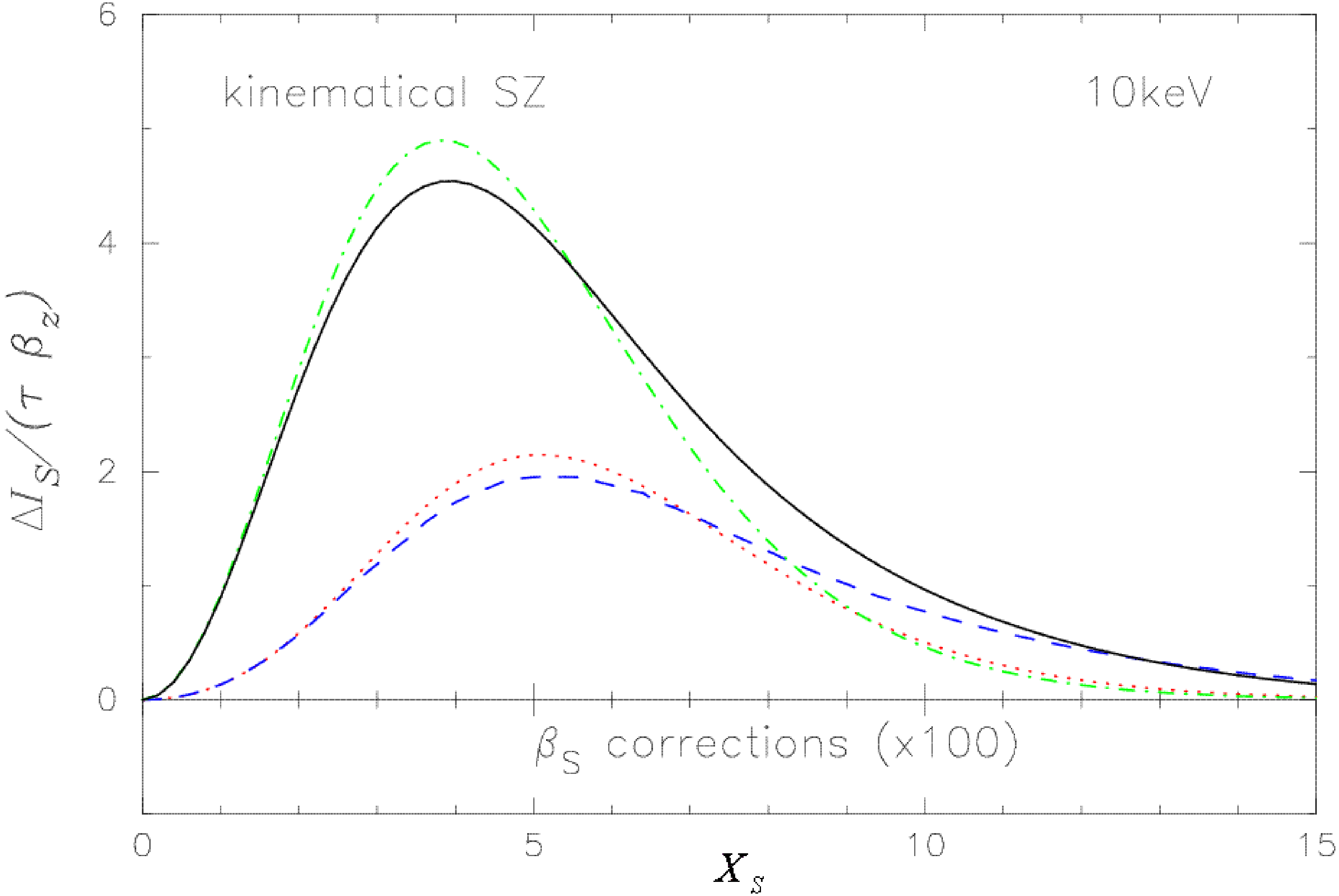
$$\vec{\beta}=\left(\frac{\vec{\beta}'\cdot\hat{\beta}_S+\beta_S}{1+\vec{\beta}'\cdot\vec{\beta}_S}\right)\hat{\beta}_S+\frac{\vec{\beta}'-(\vec{\beta}'\cdot\hat{\beta}_S)\hat{\beta}_S}{\gamma_S(1+\vec{\beta}'\cdot\vec{\beta}_S)}$$

$$\beta_z=\vec{\beta}\cdot\hat{k}$$

Thermal SZ (Effect of observer's motion)



Kinematical SZ (Effect of observer's motion)



Summary for kinematical SZ effect (Motion of the observer)

$$v_S = 372\text{km/s}$$

$$\beta_S = 1.24 \times 10^{-3}$$

For thermal SZ

β_S corrections are small

0.5% to 1% of the thermal SZ at $k_B T_e$=10keV

For kinematical SZ

β_S corrections are small

0.5% to 1% of the thermal SZ at $k_B T_e$=10keV

(c) Polarization SZ Effect

Itoh, Nozawa & Kohyama ApJ, 533, 588 (2000)

Polarized photons with Lorentz boosted Boltzman equation

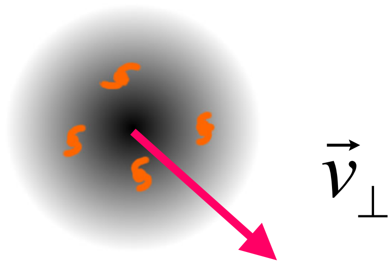
$$\frac{\partial n(\omega)}{\partial t} = -2\int \frac{d^3 p}{(2\pi)^3} d^3 p' d^3 k' \, W \, \left[1 + \vec{\eta}\cdot\vec{\varsigma}\right]$$

$$\{n(\omega)[1+n(\omega')]f(E) - n(\omega')[1+n(\omega)]f(E')\},$$

W : unpolarized crosssection

$\vec{\eta} = (\eta_1, \eta_2, \eta_3)$: Stokes parameter

$\vec{\varsigma} = (\varsigma_1, \varsigma_2, \varsigma_3)$: polarization crossection



Magnitude of the polarization of CMB

$$P = \frac{\tau \theta_e X e^X}{e^X - 1} \beta_\perp^2 \left(F_0 + \theta_e F_1 + \theta_e^2 F_2 \right),$$

$$\tau \equiv \sigma_T \int dl Ne \ , \ \ \theta_e = \frac{k_B Te}{m_e c^2}$$

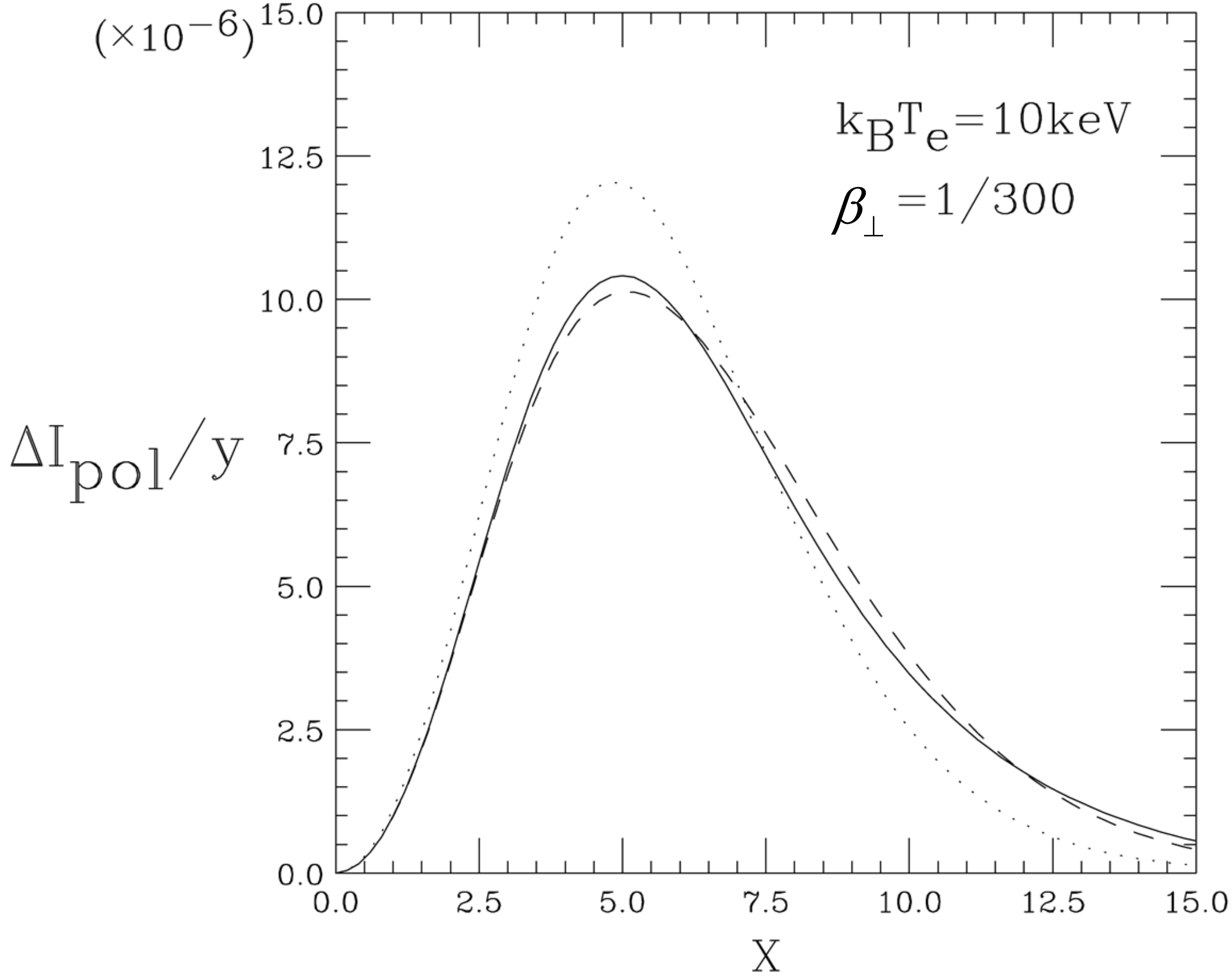
$$F_0 = \frac{1}{20} \tilde{X} \ ,$$

$$F_1 = \frac{3}{10} \tilde{X} - \frac{2}{5} \left(\tilde{X}^2 + \frac{1}{2} \tilde{S}^2 \right) + \frac{1}{20} \left(\tilde{X}^3 + 2 \tilde{X} \tilde{S}^2 \right),$$

$$F_2 = \frac{3}{4} \tilde{X} - \frac{21}{5} \left(\tilde{X}^2 + \frac{1}{2} \tilde{S}^2 \right) + \frac{867}{280} \left(\tilde{X}^3 + 2 \tilde{X} \tilde{S}^2 \right)$$
$$- \frac{4}{7} \left(\tilde{X}^4 + \frac{11}{2} \tilde{X}^2 \tilde{S}^2 + \tilde{S}^4 \right) + \frac{1}{35} \left(\tilde{X}^5 + 13 \tilde{X}^3 \tilde{S}^2 + \frac{17}{2} \tilde{X} \tilde{S}^4 \right)$$

Relativistic corrections

Distortion of the spectral intensity



Summary for Polarization SZ effect

- Relativistic corrections are very important for high-temperature CG T_e> 10keV.
- Distortion of the spectral intensity is extremely small.

$$\text{For } k_B T_e = 10\text{keV}, \; v = 1{,}000\text{km/s}$$

$$\Delta I_{pol} / y = 1 \times 10^{-5}$$

(d) Double Scattering Effect

Assumption：initial photon distribution is isotropic

$$n(X) = n_1(X) \equiv n_0(X) + \Delta n(X)$$

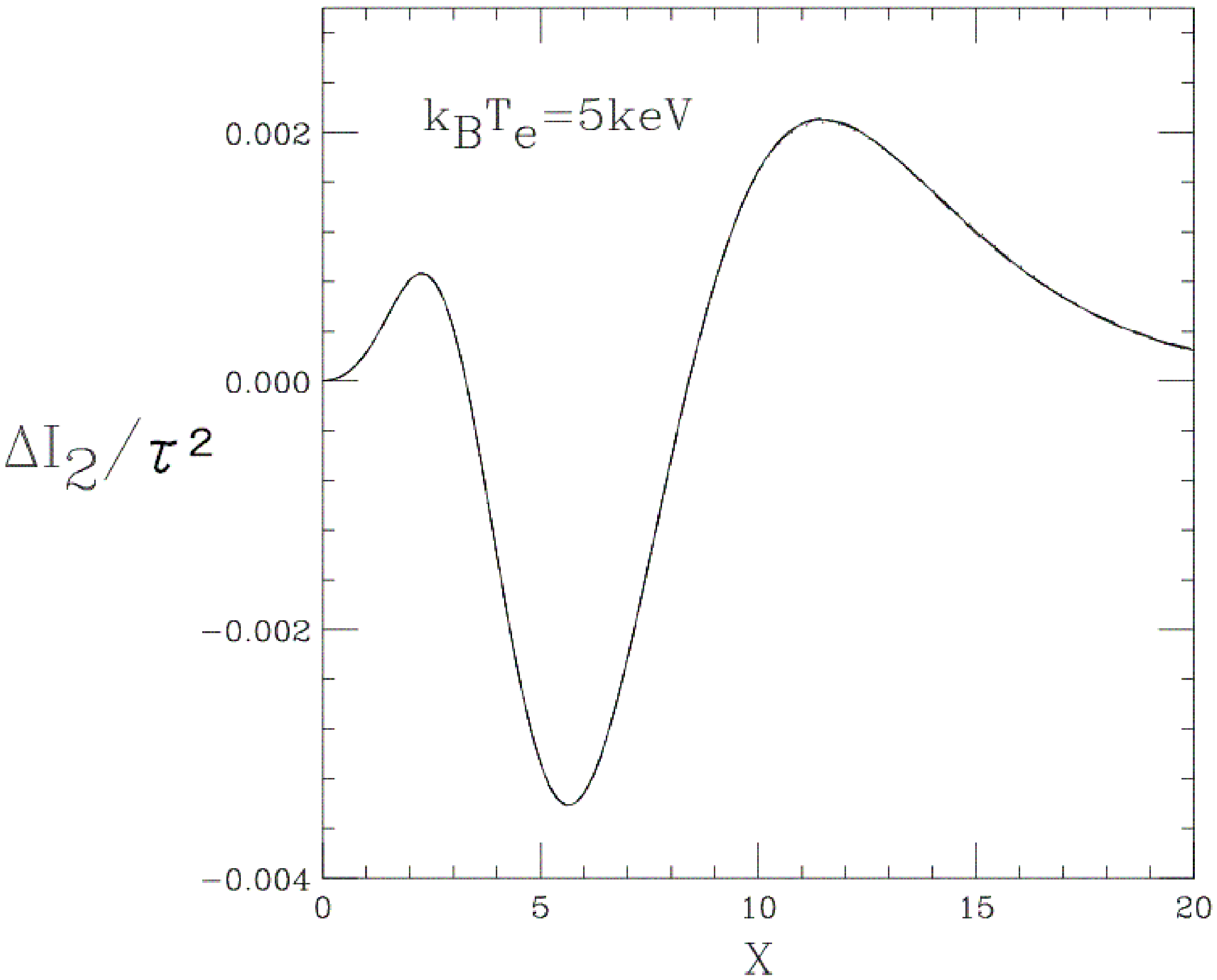
$$= n_0(X)\left[\, 1 + \boxed{\frac{\Delta n(X)}{n_0(X)}}\,\right] \qquad \text{———————} \qquad (1)$$

$$\frac{\Delta n(X)}{n_0(X)} = \frac{\tau\theta_e X e^X}{e^X - 1}[Y_0 + \theta_e Y_1 + \theta_e^2 Y_2 + \theta_e^3 Y_3 + \theta_e^4 Y_4]$$

Inserting equation (1) into Boltzman equation:

$$\Delta I = \Delta I_1 + \Delta I_2$$

$$\Delta I_2 = \frac{1}{2}\frac{\tau^2 \theta_e{}^2 X^4 e^X}{(e^X - 1)^2}(Z_0 + \theta_e Z_1 + \theta_e{}^2 Z_2 + \cdots)$$



$\Delta I_2 / \Delta I_1 \approx -0.2$ per cent for $T_e = 15\text{keV}$

Double scattering effect is safely neglected.

(3) Summary

- We have presented a relativistically covariant formalism for the SZ effects.
- With the formalism one can treat
  thermal SZ effect,
  kinematical SZ effect (CG and observer),
  polarization SZ effect, and
  multiple scattering effect
  in an unified manner.
- Relativistic corrections are very important for high-temperature CG T_e> 10keV for all SZ effects.