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ASTRONOMY AND ASTROPHYSICS

Comptonization of X-rays in Plasma Clouds. Typical Radiation Spectra

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Summary. A cloud of completely ionized plasma with sources of photons distributed in it is considered. The photon diffusion problem is solved and the distribution of the photons over their escape times is found to be similar to the light curve for X-ray bursters.

The solution of the stationary Kompaneets equation is given by the Whitteker function. The formation of radiation spectra due to comptonization in both cold and hot electron plasma clouds are obtained. The formulae obtained allow the plasma temperature in the region of the main energy release in Cyg X–I source to be determined.

The effect of Comptonization on the X-ray iron spectral line profiles strongly depends on the law of photon sources distribution over the plasma cloud.

Key words: comptonization – radiative transfer – X-ray sources

I. Introduction

1. The Formulation of the Problem

Let us consider a spherical plasma cloud of radius R, with electron density N_e and temperature T_e. The plasma is completely ionized. The plasma and the radiation interact only via Compton scattering. The optical depth of the cloud with respect to Thomson scattering is $\tau_0 \gg 1$. There is a source of photons in the center of the cloud. This is the simplest geometry of the problem. Later we shall also consider the cases where the source of photons is distributed over the cloud volume and the cloud is non-spherical.

The First Problem

At the moment $t=0$, a flare of the central source occurs. We consider the flash to be instantaneous. Therefore the time dependence of the central source luminosity is given by delta function $\delta(t)$. Solving the problem of photon diffusion in the cloud one can obtain the distribution $P(t)$ of the photons over their time of escape from the cloud. Obviously the problem is of interest for the interpretation of the observations of bursters – the sources of short X-ray bursts (Lewin and Joss, 1977). If the characteristic time of energy release in the source is quite small the time dependence of the radiation flux is determined by the time delay due to the photon diffusion in the plasma cloud surrounding the source.

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The Second Problem

It is known that multiple scatterings of X-ray photons on thermal electrons result in comptonization of the radiation – that is in distortion of the initial spectrum. In each scattering the frequency of the photon changes due to the Doppler and recoil effects. If $kT_e \gg h\nu$ the average energy of the photons increases after the scattering $\Delta\nu/\nu \approx kT_e/m_ec^2$. If $kT_e \ll h\nu$ the recoil effect results in decrease of the photon energy $\Delta\nu/\nu \approx -h\nu/m_ec^2$.

The Kompaneets equation enables one to find the solution of the problem: the time evolution of a given initial radiation spectrum due to comptonization in a homogeneous infinite medium with an electron density N_e and temperature kT_e. Solving the Kompaneets equation and giving the initial spectrum $I_0(\nu, t=0)$ the spectrum shape $I(\nu, t)$ can be found at any given time or after any given number of scatterings. Unfortunately, in astrophysics, the problem of a homogeneous infinite medium is of immediate interest to cosmology only.

The Third Problem

Again let us consider an instantaneous $\delta(t)$ source of photons with given spectrum $I_0(\nu, t=0)$ in a plasma cloud of a given geometry. The function $\Psi(\nu, t) = I(\nu, t)P(t)$ evidently gives the time dependence of the spectrum of the radiation emerging from the cloud. In application to the bursters problem we obtain the radiation spectrum evolution during an X-ray burst.

The Fourth Problem

It is suggested to search for a stationary spectrum of the radiation escaping from the plasma cloud using the results of the first and the second problems. It is assumed that stationary sources of photons exist in the cloud. Indeed, from the solution of the first problem we know the fraction of the photons that have spent a time t in the source. At the same time the solution of the second problem gives the spectrum of the photons subjected to $u = \sigma_T N_e ct$ scatterings. Obviously (see for example Chaplin and Stevens, 1973; Miyamoto, 1978) it is the convolution

$$F(\nu) = \int_0^\infty I(\nu, t)P(t)dt \tag{1}$$

that gives the stationary spectrum of the radiation escaping from the plasma cloud. Formula (1) is correct only in the case where the role of induced scattering is negligible.

From the astrophysical point of view the fourth problem is of special interest, because:

a) X-ray iron lines have been discovered recently in Her X–1 (Pravdo et al., 1977) and in Cyg X–3 (Kestenbaum et al., 1977). The photons are generated in the hot plasma or in the cold atmosphere of the normal star, irradiated by hard X-ray radiation of the binary X-ray source. Then they pass through the plasma surrounding the photon-line source (see the discussion of the astrophysical situation in Felten et al., 1972; Basko et al., 1974; Pozdnyakov et al., 1979a; Basko, 1978). Comptonization may considerably change the line profile. In the next section we shall present the result: an analytic formula for the profile of the line (locally a monochromatic line $F_0(\nu)=\delta(\nu-\nu_0)$ is emitted) escaping from the plasma cloud with the given τ_0 and T_e.

b) It enables one to analyze the change of the spectrum of the soft X-ray (or even optical or infrared) radiation passing through a hot plasma cloud with $kT_e \gg h\nu$. The difference in the photon escape times leads to a difference in the number of photon scatterings and this may give rise to the formation of a hard power law X-ray spectrum (Katz, 1976; Shapiro et al., 1976; Pozdnyakov et al., 1976, 1977, 1979b). It was mentioned by Eardley that this mechanism is similar to the well-known Fermi cosmic rays acceleration mechanism.

This problem is of particular interest in the case of Cyg X–1 where the power law spectrum of hard X-ray radiation ($3\div 80$ keV) is observed. It is likely that this mechanism is also responsible for the hard X-ray power-law spectra of quasars and nuclei of active galaxies (Katz, 1976).

c) The solution of the fourth problem enables one to determine the spectrum distortion of the hard X-ray radiation, passing through a "cold" $kT_e \ll h\nu$ plasma cloud, surrounding the central source. Such a situation probably takes place in the case of Cyg X–3.

2. *Summary of Well-known Results*

We would like to give some background before presenting new results. The equation

$$\frac{\partial n}{\partial y}=\frac{1}{x^2}\frac{\partial}{\partial x}x^4\left(n+n^2+\frac{\partial n}{\partial x}\right) \tag{2}$$

describing Compton interaction of the radiation ($h\nu \ll m_ec^2$) with the matter ($kT_e \ll m_ec^2$) was published by Kompaneets in 1956. Here $n=c^2I_\nu/8\pi h\nu^3$ is the photon occupation number in the phase-space, I_ν is the radiative intensity, $x=h\nu/kT_e$ is the dimensionless frequency and $y=\frac{kT_e}{m_ec^2}\sigma_T N_e ct=\frac{kT_e}{m_ec^2}u$ is the comptonization parameter.

The Solution of the Second Problem

If the terms n and n^2 in brackets in Kompaneets' equation are neglected, it will describe only a Doppler photon-frequency change due to scattering. Zeldovich and Sunyaev (1969) found the Cauchy-problem solution for the resulting diffusion equation

$$n(x,y)=\frac{1}{\sqrt{4\pi y}}\int_0^\infty n(z)\exp\left\{-\frac{(\ln x+3y-\ln z)^2}{4y}\right\}\frac{dz}{z} \tag{3}$$

where $n(z)$ is the arbitrary initial spectrum. If $I_\nu(x,t=0)=Ax_0\delta(x-x_0)$, then

$$I_\nu(x,y)=\frac{A}{\sqrt{4\pi y}}\exp\left\{-\frac{(\ln x_0+3y-\ln x)^2}{4y}\right\}. \tag{4}$$

If the terms n^2 and $\frac{\partial n}{\partial x}$ in the Kompaneets equation are neglected it describes the spectrum change due to the recoil effect. For this limiting case the equation has been solved by Illarionov and Sunyaev (1972a) and Arons (1972). The quantity $n\nu^4$ is constant along the characteristics

$$\frac{d\nu}{dV}=-\nu^2,\quad dV=\frac{h}{m_ec}\sigma_T N_e dt.$$

The characteristics may also be written as $\frac{d\lambda}{dt}=\sigma_T N_e h/m_e$ or $\frac{d\lambda}{du}=h/m_ec$.

Kompaneets (1956) has solved the Cauchy problem for Eq. (2) neglecting the term proportional to n^2 in brackets. The solution is given by the Whittaker functions of imaginary index.

The Solution of the Fourth Problem

Illarionov and Sunyaev (1972a) have suggested to estimate the role of Compton effect in the spatial problem using the solution for the homogeneous infinite problem. They substituted the time t in the solution of the second problem with the effective time $R\tau_0/c$ of photon escape from the cloud. Thus $y=\frac{kT_e}{m_ec^2}\tau_0^2$ and $z=\frac{h\nu}{m_ec^2}\tau_0^2$. Miyamoto (1978) have calculated numerically the photon escape time distribution as well as the influence of comptonization upon the spectrum. Katz (1976) solved numerically the full Fokker-Planck equation which describes both the space diffusion of the photons and spectrum evolution due to comptonization. Shapiro et al. (1975) have found most important asymptotics of the 4b problem (see the discussion below) solving analytically the approximate stationary Kompaneets Eq. (13). We shall show below that the solution of the stationary Kompaneets Eq. (13) is given by the Whittaker functions with real indices.

We shall present later the analytic formulae and diagrams for the solutions of the problems 1, 3, 4a, 4b, 4c, obtained under the assumption of a simple geometry of the plasma cloud and a simplest distribution of the sources of soft photons in the cloud. The results of Monte Carlo computations (Pozdnyakov et al., 1976, 1977, 1979a, 1979b) stimulated us and helped us in the analytical solution of the problem. Monte Carlo calculations have also been done by Angel (1969), Loh and Garmire (1971) and Ross et al. (1978).

We must mention also a more simple effect. Even the coherent Thomson scatterings changes the free-free radiation spectrum from the plasma cloud with $\tau_0 \gg 1$, if Thomson opacity exceeds the free-free opacity (Zeldovich and Shakura, 1969; Shakura, 1972; Felten and Rees, 1972; Illarionov and Sunyaev, 1972a, b).

II. The Distribution of Photons over the Escape Time

1. *A Homogeneous Sphere*

The solution of the problem is presented in Appendix A. If the source of photons is situated *in the center of the sphere*, the function $P(t)$ is as shown in Fig. 1. It is convenient to introduce dimensionless time $u=\sigma_T N_e ct$, characterizing the number of collisions experienced by a photon in the cloud. In the diffusion problem $u \gg 1$ and it may be regarded as a continuous variable rather than a discrete parameter. The average photon escape time is $t_0=\int_0^\infty tP(t)dt=R\tau_0/2c$, the average number of scatterings being $\bar{u}=\tau_0^2/2$. The

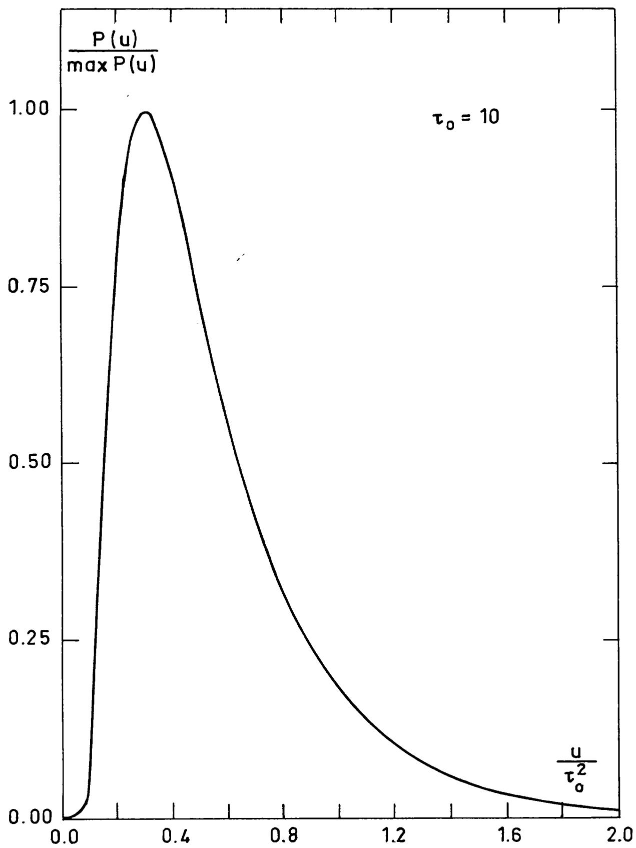


Fig. 1. The distribution of the photons over the escape time for a spherical plasma cloud. The source of the photons is in the center of the cloud. At $t=0$ the source gives an instantaneous radiation flare. The dimensionless time $u=\sigma_T N_e ct$ characterizes the number of scatterings experienced by a photon in the plasma cloud. The curve is computed for the case of the optical depth $\tau_0=10$, with respect to the Thompson scattering

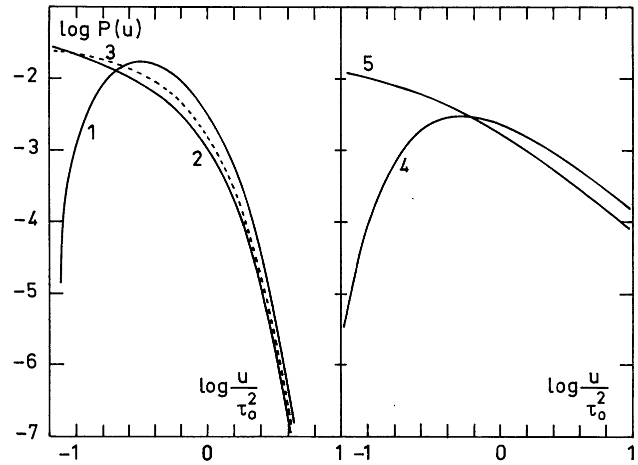


Fig. 2. The distribution of the photons over the escape time. **a** *a spherical plasma cloud* with $\tau_0=10$; curve 1 – central position of the source of photons; 2 – the sources of photons are uniformly distributed over the cloud; 3 – the sources are distributed according to the law (8). **b** *semi-infinite medium:* 4 – the sources are at the depth $\tau_0=10$; 5 – the sources are uniformly distributed in the zone $0 \leqq \tau \leqq \tau_0$

peak of $P(t)$ lies near $t=0.3R\tau_0/c$ or $u_0=0.3\tau_0^2$. When $u \gg u_0$ we have asymptotics

$$P_1(u)=\frac{2\pi^2}{3(\tau_0+\frac{2}{3})^2}\exp\left\{-\frac{u\pi^2}{3(\tau_0+\frac{2}{3})^2}\right\} \tag{5}$$

and when $1 \ll u \ll u_0$

$$P_2(u)=\frac{3\sqrt{3}}{4\sqrt{\pi}}\frac{\tau_0^2}{u^{3/2}}\exp\left\{-\frac{3\tau_0^2}{4u}\right\}. \tag{6}$$

The function $P(u)$ is well described by composition of asymptotics of the kind:

$$P_3(u)=\frac{3\sqrt{3}}{4\sqrt{\pi}}\frac{\tau_0^2}{u^{3/2}}\left(1+\frac{8\pi^2}{9\tau_0}\left(\frac{u}{\tau_0^2}\right)^{3/2}\right)\exp\left\{-\frac{3\tau_0^2}{4u}-\frac{u\pi^2}{3(\tau_0+\frac{2}{3})^2}\right\}.$$

In case of *uniform distribution* of the sources of photons over the sphere volume the function $P(u)$ has the shape shown in Fig. 2. For small $1 \ll u \ll u_0$

$$P_4(u)\cong\frac{1}{\tau_0}\left(\frac{3}{\pi u}\right)^{1/2}\left(1-\frac{2}{\tau_0}\left(\frac{\pi u}{3}\right)^{1/2}\right). \tag{7}$$

For $u \gg u_0$

$$P_5(u)\cong\frac{2}{\tau_0^2}\exp\left\{-\frac{\pi^2 u}{3\tau_0^2}\right\} \tag{5a}$$

the asymptotic coincides with $P_1(u)$. It is convenient to use the composition of asymptotics

$$P_6(u)=\frac{1}{\tau_0}\sqrt{\frac{3}{\pi u}}\left(1+2\frac{\sqrt{u}}{\tau_0}\right)\exp\left\{-\frac{\pi^2 u}{3\tau_0^2}\right\}.$$

In this case $\bar{u}=\tau_0^2/5$.

Interesting is the case where the sources of photons are distributed according to the law (see Fig. 3)

$$\varphi(\tau)=\frac{\tau_0}{\pi\tau}\sin\frac{\pi\tau}{\tau_0}. \tag{8}$$

This is an intermediate case between those of uniform distribution of the sources and the central source. In this case $P(u)$ is very simple because it is an eigenfunction of the diffusion equation

$$P(u)=\frac{\pi^2}{3(\tau_0+\frac{2}{3})^2}\exp\left\{-\frac{\pi^2 u}{3(\tau_0+\frac{2}{3})^2}\right\}=\beta\exp(-\beta u) \tag{9}$$

where

$$\beta=\pi^2/3(\tau_0+\tfrac{2}{3})^2\cong\frac{3}{\tau_0^2}.$$

The average number of scatterings experienced by the photons in the source is equal to $\bar{u}=\beta^{-1}$.

2. *Inhomogeneous Density Distribution*

The case where the electron density in a spherical cloud is distributed according to the law $N_e \sim r^{-2}$, is of interest from the methodological point of view. Here the equation of photon diffusion (see formula (A.1) in Appendix) is identical to

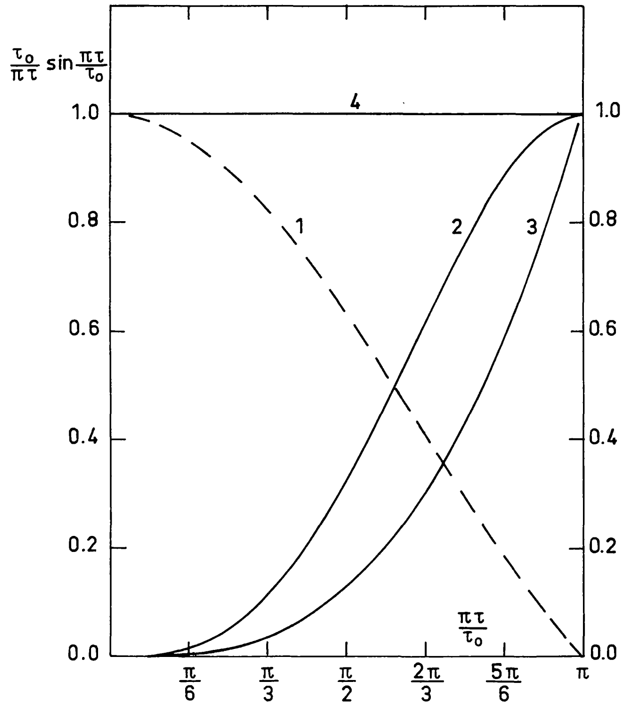


Fig. 3. Properties of the photon sources distribution (8). 1 – the dependence of the sources density on the radius or τ. 2 – the ratio of the luminosity of the sources in the zone with τ less than the one given to the total luminosity of all the sources in the cloud. For comparison the same ratio is given for the case of a uniform distribution of sources (curve 3) and for the central source (curve 4)

Kompaneets' Eq. (2) if in the latter the terms $(n+n^2)$ in brackets are neglected (which are responsible for the frequency change due to the recoil effect in spontaneous and stimulated scatterings). The solution of the Cauchy problem is determined by formula (3). Let us consider a sphere of radius r_0, to be a source of photons and a sphere of radius r_1 to be the boundary of a plasma cloud. The optical depth of the cloud is $\int_{r_0}^{\infty}\sigma_T N_e dr=\tau_0$ and $\int_{\tau_1}^{\infty}\sigma_T N_e dr=1$ $(\tau_0 \gg 1)$. The problem is formally reduced to the solution of the Cauchy problem for a semi-definite medium. Using solution (3) and the reflection method (see Appendix A) we find

$$P(u)=\tfrac{3}{2}\frac{\ln\tau_0}{\tau_0\sqrt{\pi}}\left(\frac{3\tau_0^2}{u}\right)^{3/2}\exp\left\{-\frac{3u}{4\tau_0^2}-\frac{3\tau_0^2\ln^2\tau_0}{4u}\right\}.$$

The exact solution contains both asymptotics (5) and (6) we are interested in.

3. *Semi-definite Medium*

If the sources of photons are at the depth $\tau_0 \gg 1$ in the semi-infinite homogeneous medium, then:

$$P(u)=\frac{\sqrt{3}}{2\sqrt{\pi}}\frac{\tau_0}{u^{3/2}}\exp\left\{-\frac{3\tau_0^2}{4u}\right\}. \tag{11}$$

If the sources are distributed uniformly in the layer $0 \leqq \tau \leqq \tau_0$, then

$$P(u)=\frac{1}{\tau_0\sqrt{3\pi u}}\left(1-\exp\left\{-\frac{3\tau_0^2}{4u}\right\}\right). \tag{12}$$

When $u \gg \tau_0^2$ both formulae (11) and (12) have asymptotic $P(u) \sim u^{-3/2}$.

When $u \ll \tau_0^2$ formula (12) coincides with (7). Of interest is also the case of an exponential flat atmosphere with $N_e=N_0\exp\left\{-\frac{r-r_1}{H}\right\}$ and the radius of the lower boundary surface $r_1 \gg H$. The photon sources are located at the depth $\tau_0 \ll \tau_1$ where $\tau_1=\int_{r^1}^{\infty}\sigma_T N_e dr$. If the albedo of the lower boundary surface $A=1$, then, with $u \ll \tau_0^2$, the photon escape is determined by an asymptotic of $P_2(u)$ kind. With $\tau_0^2 \ll u \ll \tau_1^2$ we obtain

$$P(u) \approx \frac{C}{u^{3/2}}\exp\{-u/3\tau_0^2\}$$

and with $u > \tau_1^2$ the asymptotic $P(u) \cong \exp\{-u/\tau_1^2\}$ takes place.

When $A=0$ the portion of photons absorbed by the lower boundary surface is equal to $\frac{1}{2}\frac{\tau_0+2/3}{\tau_1+4/3}$.

4. *Disk*

If in a homogeneous disk the sources of photons are distributed in the plane of symmetry or homogeneously over its volume, then $P(u)$ differs slightly from the approximate formulae for a spherical plasma cloud. If sources are distributed according to the eigenfunction of the diffusion equation

$$P(u)=\beta\exp(-\beta u) \quad \text{and} \quad \beta=\pi^2/12(\tau_0+2/3)^2.$$

Here τ_0 corresponds to the halfthickness of the disk.

If the surface of the disk is illuminated from the outside by an instantaneous flare, part of the photons penetrate into the disk and later, after diffusion in the disk, escape from it. Then, $P(u)=1/2(\pi u)^{1/2}$ when $1 \ll u \ll \tau_0^2$,

$$P(u)=\frac{4}{\tau_0}\beta\exp(-\beta u), \text{ when } \quad u \gg \tau_0^2.$$

5. *Cylinder*

In this case $\beta=3\pi^2/16(\tau_0+2/3)^2$.

III. Solution of the Stationary Equation of Comptonization

When the probability of photon escape from the plasma cloud is given by the simple formula $P(u)=\beta\exp(-\beta u)$ the comptonization problem is substantially simplified and may be reduced to the solution of the stationary equation (Chapline and Steevens, 1973; Shapiro et al., 1976). Let us remind that the exponential law of photon escape from a homogeneous plasma cloud holds only when the sources of photons in the cloud are distributed according to law (8).

Let us write out the stationary equation which is in fact the equation for the convolution $N(x)=\int_0^{\infty} n(x,u)P(u)du$ with $P(u)$ given by (9). For this purpose, we shall integrate Kompaneets' Eq. (2) (without the term responsible for the induced scattering) preliminary multiplied by $\beta\exp(-\beta u)$

$$\int_0^{\infty}\beta\exp(-\beta u)\frac{\partial n}{\partial u}du=\frac{a}{x^2}\frac{\partial}{\partial x}x^4 \cdot\left(\int_0^{\infty}\beta n\exp(-\beta u)du+\frac{\partial}{\partial x}\int_0^{\infty}\beta n\exp(-\beta u)\,du\right.$$

$$a=\frac{kT_e}{m_e c^2}.$$

Integration by parts of the left part of the equation taking into account the initial condition $n(x,0)=f(x)/x^3$ yields:

$$\frac{1}{x^2}\frac{d}{dx}x^4\left(\frac{dN}{dx}+N\right)=\gamma N-\gamma f(x)/x^3 \tag{13}$$

where

$$\gamma=\beta/a=\pi^2 m_e c^2/3(\tau_0+\tfrac{2}{3})^2 kT_e, \qquad a=\frac{kT_e}{m_e c^2}. \tag{14}$$

In this equation $f(x)$ determines the radiation spectrum of the photon sources, their spatial distribution $\varphi(\tau)$ corresponds to (8). The plasma is assumed to be isothermal. In the Kompaneets' Eq. (2) and in Eq. (13) the term $\frac{dN}{dx}$ in brackets describes the energy change of the photon due to the Doppler-effect. The term N in brackets describes the energy change due to the recoil effect.

Equation (13) can be solved easily in two limit cases:

a) when $kT_e \gg h\nu$, i.e. $x \ll 1$ and one can neglect the recoil effect as compared with that of the Doppler effect;

b) vice versa when $h\nu \gg kT_e$ and the recoil effect is more important. We shall find below the solutions of Eq. (13) in these simple cases, then the solution of the full Eq. (13) taking into account both Doppler and recoil effects. This solution is conveyed by the Whittaker function.

1. $kT_e \gg h\nu$; the Spectrum of the Escaping Radiation

If the recoil effect is neglected then from (13) we obtain the equation

$$\frac{1}{x^2}\frac{d}{dx}x^4\frac{dN}{dx}-\gamma N+\gamma f(x)/x^3=0 \tag{15}$$

taking only into account the Doppler change of the photon frequency. If the source spectrum is represented by the δ-function $f(x)=x_0\delta(x-x_0)$ its solution $F(x,x_0)$ is

$$\begin{aligned} F_1(x,x_0)&=x^3N(x,x_0)=\frac{\alpha(\alpha+3)}{(2\alpha+3)}\left(\frac{x}{x_0}\right)^{\alpha+3} && \text{when} \quad 0\leqq x\leqq x_0 \\ F_2(x,x_0)&=\frac{\alpha(\alpha+3)}{(2\alpha+3)}\left(\frac{x}{x_0}\right)^{-\alpha} && \text{when} \quad x_0\leqq x<\infty . \end{aligned} \tag{16}$$

The number of photons is normalized to the unity

$$\int_0^\infty f(x)\frac{dx}{x}=\int_0^\infty F(x)\frac{dx}{x}=1 .$$

This solution describes the distortion of the monochromatic line spectrum due to scattering on hot electrons. The radiation spectrum is power law in both wings of the line.

The spectral index in the high frequency wing of the spectrum is equal to

$$\alpha=(\tfrac{9}{4}+\gamma)^{1/2}-\tfrac{3}{2} \tag{17}$$

(Shapiro et al., 1976). In Sect. III.4. it is shown that for a source with the Planckian spectrum with $T_e \ll T_e$ the far wing $kT_r \ll h\nu \ll kT_e$ of the spectrum of the radiation emerging from the cloud has the same shape as (16).

Monte-Carlo calculations (Pozdnyakov et al., 1979b) have shown that the resulting spectrum in the problem with $\tau_0=3$ and $kT_e=50$ and 100 keV on the range $x_0<x<1$ has the spectral indices coinciding with (17) to an accuracy of several percent. In the case of small $\tau_0<3$ the spectral indices differ drastically from (17). There is also great discrepancy between the results of the Monte-Carlo calculation and the approximation in the problem with $kT_e=250$ keV and $\tau_0=3$. Let us remind that Pozdnyakov et al. (1977) have obtained for the spectral index α an analytic formula that is true in the limit $\tau_0<1$, $kT_e \gg m_e c^2$

$$\alpha=-\lg\tau_0/\lg 12\left(\frac{kT_e}{m_e c^2}\right)^2$$

and Pozdnyakov et al. (1979b) have given an approximation formula

$$\alpha=(-\lg\tau_0+2/(a+3))/\lg(12a^2+25a), \qquad a=\frac{kT_e}{m_e c^2}$$

which well describes the results of Monte-Carlo calculations when $kT_e>50$ keV and $\tau_0\leqq 3$.

2. $kT_e \gg h\nu$; Energy Loss by Electrons

It has been shown that in the infinite homogeneous medium the radiation energy density increases with time according to the law $\varepsilon_r=\varepsilon_0\exp(4y)$ (Kompaneets. 1956). This law is correct only when the Doppler effect dominates. The same method is used to multiply Eq. (13) by x^3 and integrate it over x. As a result, we find that when the luminosity of the sources of photons is L_0 the total luminosity of the plasma cloud is

$$L=L_0\gamma/(\gamma-4)=L_0\frac{\alpha(\alpha+3)}{(\alpha-1)(\alpha+4)}. \tag{18}$$

The solution is true only when $\gamma \gg 4$, $\alpha \gg 1$ (see the discussion in Sect. III.5.).

3. $h\nu \gg kT_e$; the Shape of the Radiation Spectrum

In the ultimate case of low temperature plasma and sources of hard photons in the cloud the problem is reduced to the solution of the following equation:

$$\frac{1}{z^2}\frac{d}{dz}z^4N-\beta N=-\beta f(z)/z^3 . \tag{19}$$

In this approximation Doppler effect does not influence the spectrum, therefore the resulting emission spectrum is independent of electron temperature. It is suitable to introduce a dimensionless variable $z=h\nu/m_e c^2$ instead of $x=h\nu/kT_e$.

The solution of Eq. (19) is as follows:

$$F_\nu(z)=\frac{\beta}{z}\exp(-\beta/z)\int_z^\infty f(\xi)\exp(\beta/\xi)\frac{d\xi}{\xi}. \tag{20}$$

With $f(z)=z_0\delta(z-z_0)$ we find how the recoil effect influences a monochromatic line profile

$$F_\nu(z)=\frac{\beta}{z}\exp\left\{-\beta\left(\frac{1}{z}-\frac{1}{z_0}\right)\right\}\Theta(z_0-z) \tag{20a}$$

where

$$\Theta(z_0-z)=\begin{cases}0 & \text{with} \quad z>z_0\\ 1 & \text{with} \quad z<z_0\end{cases}$$

is the Θ-function,

$$\frac{\beta}{z}=\frac{\pi^2 m_e c^2}{3(\tau_0+\frac{2}{3})^2 h\nu}.$$

For a power law spectrum of the sources $f(z)=Az^{-\alpha}$ the integral in (20) can be easily found in case $\alpha \geqq 1$. Two simplest examples with $\alpha=1$ are given:

$$F_\nu(z)=\frac{A}{z}[1-\exp(-\beta/z)] \xrightarrow[z \gg \beta]{} \frac{A\beta}{z^2}$$

and $\alpha=2$

$$F_\nu(z)=\frac{A}{z^2}\left[1-\frac{z}{\beta}(1-\exp\{-\beta/z\})\right] \xrightarrow[z \gg \beta]{} \frac{A\beta}{2z^3}.$$

The integral form of the solution gives evidence that with $z>\beta$ that is $\frac{h\nu}{m_e c^2}\tau_0^2>3$, the spectral index increases by unity[1] $\alpha \to \alpha+1$. Actually with $z \gg \beta$ the exponents $\exp\{-\beta/\xi\}$ and $\exp\{-\beta/z\}$ in (20) are close to unity and with $\alpha>0$ we have

$$F_\nu(z)=\frac{A\beta}{\alpha} z^{-\alpha-1}. \tag{21}$$

For $0<\alpha<1$ simultaneously with intensity decrease in the range $z>\beta$, intensity increases in the vicinity of $z \sim \beta$. Indeed integration by parts of the solution (20) easily gives the first corrections for the spectrum $f(z)=Az^{-\alpha}$, with small $z \ll \beta$

$$F_\nu(z)=Az^{-\alpha}\left[1+(1-\alpha)\frac{z}{\beta}+(1-\alpha)(2-\alpha)\frac{z^2}{\beta^2}+\dots\right].$$

Comparison with (21) shows that near $z \sim \beta$ an increase of intensity occurs. It is only natural because the number of photons is conserved. Similar analysis can be carried out for a source spectrum of the following form $f(z)=A\exp\{-h\nu/kT_0\}$. With large $z>\beta$, that is $\frac{h\nu}{m_e c^2}\tau_0^2>3$, the resulting spectrum is an exponential integral. Maximum intensity occurs in the vicinity of $z \sim \beta$. The scattering does not influence the spectrum shape in the range $z \ll \beta$. In Sect. IV the dependence of escape radiation spectrum on the spatial distribution of photon sources in the plasma cloud will be discussed.

4. Emission Spectrum (General Case)

The stationary Kompaneets Eq. (13) that takes into account both the recoil and Doppler effects, by the substitution of a new variable $N=x^2\exp(-x/2)W$, is reduced to the well known equation

$$\frac{d^2W}{dx^2}+\left(-\tfrac{1}{4}+\frac{2}{x}+\frac{1/4-(9/4+\gamma)}{x^2}\right)W=-\gamma f(x)\exp(-x/2). \tag{22}$$

The case where $f(x)=0$ has been investigated by Whittaker (see for example Whittaker and Watson, 1963). Its solution are the Whittaker functions $W_{2,(9/4+\gamma)^{1/2}}(x)$ and $M_{2,(9/4+\gamma)^{1/2}}(x)$. They have convenient integral representations. In Appendix B the exact solution of the boundary problem is given for Eq. (22). It enables us (using integral representations of the Whittaker functions) to calculate numerically the spectrum of the radiation emerging from an isothermal plasma cloud. The problem is reduced to the calculation of one integral. This is a much easier procedure than the direct numerical solution of Eq. (13). If $f(x)=x_0\delta(x-x_0)$ and $x_0 \ll 1$ the spectrum of the photons emerging from the cloud is described with the simple formula:

$$\begin{aligned} F_\nu(x,x_0)&=\frac{\alpha(\alpha+3)}{2\alpha+3}\left(\frac{x}{x_0}\right)^{3+\alpha} \quad \text{when} \quad 0 \leqq x \leqq x_0 \\ F_\nu(x)&=B(\gamma,x_0)x\exp(-x/2)W_{2,(9/4+\gamma)^{1/2}}(x) \end{aligned} \tag{23}$$

when $x \geqq x_0$.
Where

$$B(\gamma,x)=\frac{\gamma\Gamma((\frac{9}{4}+\gamma)^{1/2}-\frac{3}{2})}{\Gamma((9+4\gamma)^{1/2})(9+4\gamma)^{1/2}}x_0^{(9/4+\gamma)^{1/2}-3/2}=\frac{\alpha(\alpha+3)\Gamma(\alpha)x_0^\alpha}{\Gamma(2\alpha+4)}$$

and $\Gamma(z)$ is the gamma function. The integral representation is given as:

$$W_{2,(9/4+\gamma)^{1/2}}(x)=\frac{x^2\exp(-x/2)}{\Gamma((\frac{9}{4}+\gamma)^{1/2}-\frac{3}{2})}\int_0^\infty t^{(9/4+\gamma)^{1/2}-5/2} \cdot\exp(-t)\left(1+\frac{t}{x}\right)^{(9/4+\gamma)^{1/2}+3/2}dt. \tag{24}$$

For $x \geqq x_0$ is follows from (23) and (24) that

$$\begin{aligned} F_\nu(x)&=\frac{\alpha(\alpha+3)x_0^\alpha}{\Gamma(2\alpha+4)}x^{-\alpha}\exp(-x)\int_0^\infty t^{\alpha-1}\exp(-t)(x+t)^{\alpha+3}dt \\ &=\frac{\alpha(\alpha+3)x_0^\alpha}{\Gamma(2\alpha+4)}x^3\exp(-x)\int_0^\infty t^{\alpha-1}\exp(-t)\left(1+\frac{t}{x}\right)^{\alpha+3}dt. \end{aligned} \tag{23a}$$

In the case of small $x \ll 1$ from this integral representation it is easy to derive power law (16) describing the high frequency wing of the spectrum, which is formed due to the Doppler effect. In the case $x \gg 1$ we obtain Wien distribution

$$F_\nu(x)=B(\gamma,x_0)x^3\exp(-x) \tag{23b}$$

When $\alpha+3$ is a positive integer the expression $\left(1+\frac{t}{x}\right)^{\alpha+3}$ is a polynomial.

In such cases expressions for high-frequency wings in the escaping radiation spectra are series with a finite number of terms: when $\alpha=1$, $\gamma=4$

$$F_\nu(x,x_0)=\frac{4x_0}{5x}\exp(-x)(1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\tfrac{1}{24}x^4). \tag{25}$$

When $\alpha=2$, $\gamma=10$

$$F_\nu(x,x_0)=\frac{10x_0^2}{7x^2}\exp(-x)(1+\tfrac{5}{6}x+\tfrac{1}{3}x^2+\tfrac{1}{12}x^3+\tfrac{1}{72}x^4+\tfrac{1}{720}x^5)$$

and so on.

The most interesting solution (25) for the case $\gamma=4$ has been found by Shapiro et al. (1976). The Wien law is involved only at a later stage (Fig. 4): deviations from the power law spectrum are noticeable only when $x>3+4$. When $\gamma \ll 1$ it is possible to obtain the expansion (23) over a small parameter γ and find

$$F_\nu(x)=\frac{x^3\exp(-x)}{2+\frac{2}{3}\gamma\left(\ln\frac{1}{x_0}+2,4\right)}\left[1+\frac{\gamma}{x^3}(\tfrac{2}{3}+x+x^2)\right].$$

With $\gamma \to 0$ we get the Wien spectrum.

With small $x_0 \ll 1$ using Table 1 it is easy to determine the radiation spectrum for the most interesting range of spectral indices $0.1 \leqq \alpha \leqq 0.9$ or the parameter $0.3 \leqq \gamma \leqq 3.5$. Until $x_0 \ll x < 0,3$ the spectrum obeys the power law and is given by

1 Similar result has been obtained independently by A. F. Illarionov

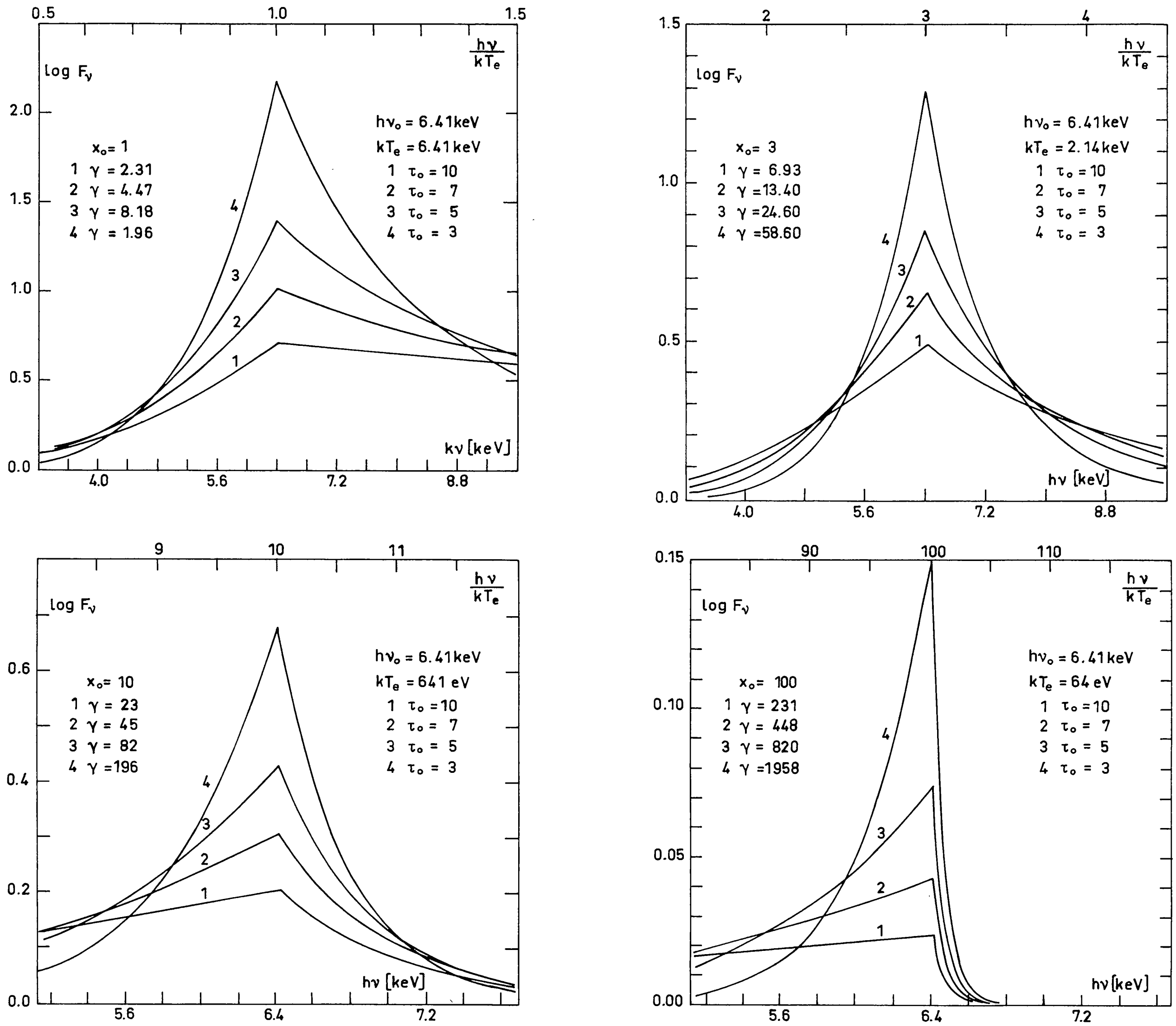


Fig. 4. The effect of comptonization on the profile of the monochromatic line escaping from the spherical plasma cloud. The sources of photons are distributed according to the law (8). The increase of $x_0 = h\nu_0/kT_e$ from 1 (Fig. 4a) to 100 in (Fig. 4d) is accompanied by decreasing Doppler-effect role and greater role of the recoil effect. The parameters on the top left hand corner of the drawings and the upper scale are in dimensionless units. For illustration the parameters, corresponding to the specific case of a weakly ionized iron K_α-line are shown to the top right and on the lower scale of the drawings. Line splitting into a doublet with $\Delta E = 13$ eV is neglected

Table 1. Emission spectrum formed by the comptonization of the low frequency photons

	α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	$x \diagdown \gamma$	0.31	0.64	0.99	1.36	1.75	2.16	2.59	3.04	3.5
$V(x,\gamma)$ $=F_\nu(x)/x_0^\alpha$	0.2	0.127	0.282	0.476	0.720	1.024	1.403	1.874	2.458	3.179
	0.6	0.175	0.289	0.405	0.523	0.646	0.772	0.907	1.048	1.197
	1.0	0.273	0.353	0.426	0.492	0.553	0.610	0.664	0.715	0.764
	1.4	0.388	0.429	0.462	0.490	0.512	0.530	0.545	0.556	0.566
	1.8	0.488	0.491	0.490	0.487	0.482	0.475	0.466	0.457	0.447
	2.5	0.586	0.538	0.495	0.457	0.423	0.392	0.363	0.337	0.312
	3.0	0.591	0.523	0.465	0.415	0.372	0.334	0.301	0.271	0.245
	4.0	0.494	0.419	0.357	0.305	0.262	0.226	0.195	0.169	0.147
	$d(\gamma)$	2.70	2.60	2.50	2.55	2.68	3.03	3.30	4.29	6.32

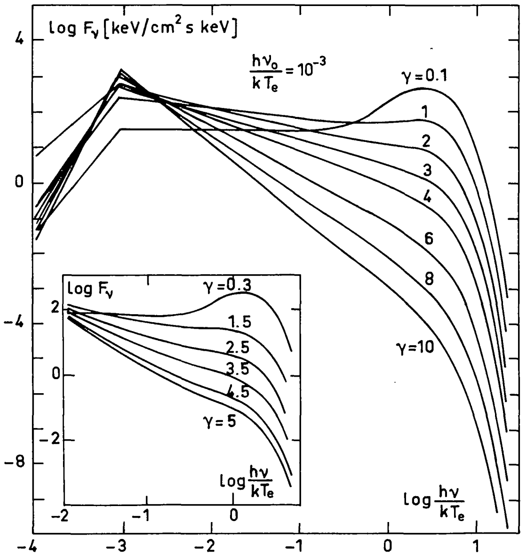


Fig. 5. The spectrum resulting from comptonization of low-frequency photons ($h\nu_0 = 10^{-3}$ kT$_e$) in a high temperature plasma clouds with different parameters γ (14)

formula (16). When $x > 5$ the spectrum is described by the Wien formula (23b). In the intermediate zone $0.3 < x < 5$ the spectrum can be derived from Table 1, where numerical values of $V(x, \gamma)$ are shown. The function $V(x, \gamma)$ is determined in the following way:

$$F_\nu(x) = x_0^\alpha V(x, \gamma).$$

The general solution of Eq. (13) given in Appendix B also allows the evolution of the spectrum with $x_0 > 1$ or $h\nu_0 > kT_e$ to be traced (Fig. 4). This solution takes into account both the recoil and Doppler effects. Figure 9b shows the comparison of this general solution with solution 20a, which neglects the Doppler-effect frequency change. The plots in Figs. 4, 8, 9 have been obtained in a diffusion approximation, therefore they do not take into account the direct escape of photons from the sources and are not adequate to describe the spectrum of photons that underwent only few scatterings in the cloud. When $\tau_0 \gg 1$ the spectra calculated describe well the line profile, and if $\tau_0 < 5$ the profiles of the lines are easily calculated using the Monte-Carlo method (Pozdnyakov et al., 1979a).

5. General Case. An Arbitrary Radiation Spectrum of Sources

In the case of an arbitrary spectrum $f(x)$ of low frequency photons the spectrum of the escaping radiation may be represented as a linear superposition

$$F_\nu(x) = \int_0^\infty \frac{1}{x_0} G(x, x_0) f(x_0) dx_0 \tag{26}$$

where the function $G(x, x_0)$ is given by formula (23) or (B5) and is a response to the monochromatic spectrum of sources (the Green function).

As an illustration let us consider a source spectrum of the type $f(x) = (a^\delta x^\delta / \Gamma(\delta) x^\delta \exp(-ax)$ where the coefficient $a = (T_e/T_r) \gg 1$ characterizes the ratio between the electron temperature and the effective radiation temperature of the photon sources in a plasma cloud. The number of photons is normalized to the unity $\int_0^\infty f(x) \frac{dx}{x} = 1$. Substituting the simple formulae (16) for $G(x, x_0)$ in (26) we obtain the resulting spectrum in the low frequency region $h\nu \ll kT_r$, $ax \ll 1$. When $\alpha + 3 > \delta$ the spectrum in the low frequency zone does not change its slope

$$F_\nu(x) = \frac{\alpha(\alpha+3)}{(3+\alpha-\delta)(\alpha+\delta)} x^\delta.$$

If $2 + \alpha > \delta$ the intensity decreases. If $2 < \delta - \alpha < 3$ it increases. In the case of the steep spectrum of the photon sources $\delta > \alpha + 3$ the low frequency tail of the emerging radiation spectrum is described by the first formula (16) similar to that in the case where the spectrum of the sources is a monochromatic line.

According to (23a) and (26) the high frequency part of the spectrum $h\nu \gg kT_r$ is identical to the high frequency tail of the monochromatic line emerging from the source

$$F_\nu(x) = \frac{\alpha(\alpha+3)\Gamma(\alpha)\Gamma(\alpha+\delta)}{\Gamma(\delta)\Gamma(2\alpha+4)a^\alpha} x \exp(-x/2) W_{2,(9/4+\gamma)^{1/2}}(x).$$

More interesting is the case when the photon sources have blackbody spectrum

$$f(x) = \frac{a^3}{2\zeta(3)} \frac{x^3}{\exp(ax) - 1}$$

with $T_r \ll T_e$ or $a = \frac{T_e}{T_r} \gg 1$. Here and below $\zeta(z)$ is the Riemann zeta function. According to (16) and (26) in the Rayleigh-Jeans part of the emerging spectrum there is a decrease in both intensity

$$F_\nu(x) = \frac{\alpha(\alpha+3)a^2}{2\zeta(3)(\alpha+1)(\alpha+2)} x^2$$

and radiation brightness temperature

$$(T_e - T_r)/T_r = -2/(\alpha+1)(\alpha+2) = -2/(\gamma+2).$$

When $\gamma \gg 2$ one has

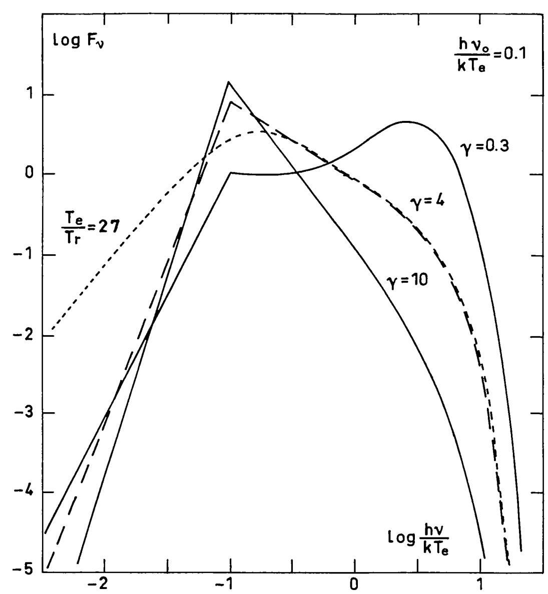
$$\Delta T_r/T_r = -\frac{6}{\pi^2} \frac{kT_e}{m_e c^2} (\tau_0 + \tfrac{2}{3})^2.$$

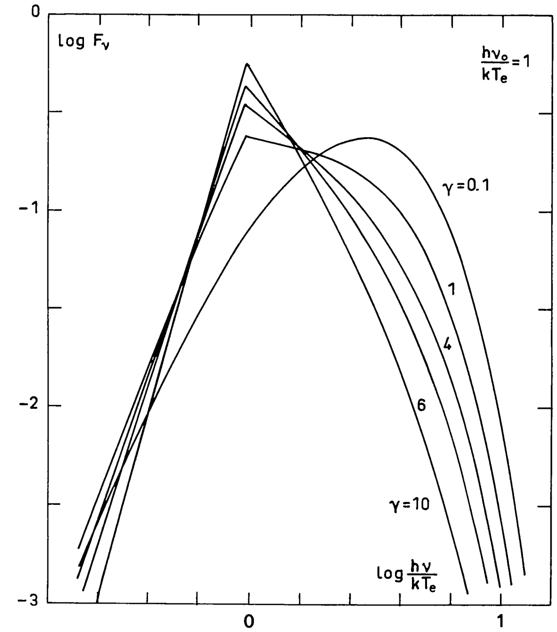
This effect exists also in a homogeneous medium. It has been mentioned earlier by Zeldovich and Sunyaev (1969). In the case of the initial Rayleigh-Jeans spectrum the intensity decreases as $\exp\{-y\}$ where

$$y = \int_0^t \frac{kT_e}{m_e c^2} \sigma_T N_e c dt.$$

The high frequency $h\nu \gg kT_r$ part of the spectrum is similar to the high frequency tail of the monochromatic line emerging from the source (Fig. 6a).

$$F_\nu = \frac{\alpha\zeta(\alpha+3)\Gamma(\alpha)\Gamma(\alpha+4)}{2\Gamma(2\alpha+4)\zeta(3)a^\alpha} x \exp(-x/2) W_{2,(9/4+\gamma)}(x).$$





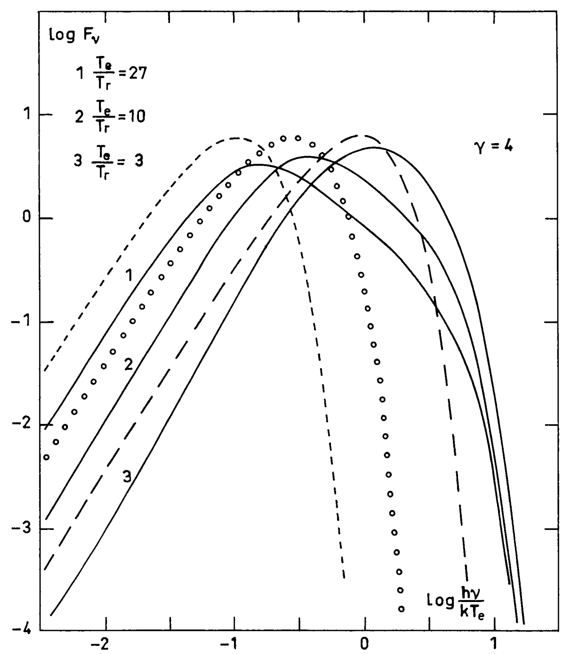


Fig. 6. The spectrum resulting from comptonization of low-frequency photons in a high temperature plasma cloud. The spectrum of the sources of low frequency photons is given as. **a** the monochromatic line with $h\nu_0=10^{-1}\,kT_e$ (for comparison the curve "p" is given, which was computed for the Planckian spectrum of the sources with $T_r=\frac{1}{27}T_e$ i.e. $\overline{h\nu}_0=0,1kT_e$. The curves are normalized to the same number of photons; **b** the monochromatic line with $h\nu_0=kT_e$; **c** Planckian spectra with different ratios T_e/T_r; The curves are normalized to the same number of photons

The Comparison with (23) shows that high frequency tails are identical if

$$h\nu_0=kT_r\left(\frac{\Gamma(3+\alpha)\zeta(3+\alpha)}{2\zeta(3)}\right)^{1/\alpha}=pkT_r.$$

The coefficient p is equal to 4.2; 3.7; 3.2; 2.7; 2.4 and 2.3 for $\alpha=4$; 3; 2; 1; $\frac{1}{2}$ and $\frac{1}{3}$ correspondingly.

6. The Luminosity of the Source

In Sect. III.2 the convenient formula $\frac{L}{L_0}=\frac{\gamma}{\gamma-4}$ is derived, that relates the luminosity of the sources of low frequency emission L_0 with the total luminosity of the plasma cloud L (the rate of the energy loss by all the electrons in the cloud is equal to $L-L_0$). Evidently the formula no longer holds when $\gamma\to 4$. Let us find the limits within which the formula is applied.

We have a simple analytic solution (25) for the case $\gamma=4$ and it is easy to integrate it over the frequency when $x_0\ll 1$.

$$\frac{L}{L_0}\approx\frac{4}{5}\left(\ln\frac{1}{x_0}+1,5\right).$$

It is clear now that formula (18) is true until

$$\frac{\gamma}{\gamma-4}\ll\frac{4}{5}\left(\ln\frac{1}{x_0}+1,5\right).$$

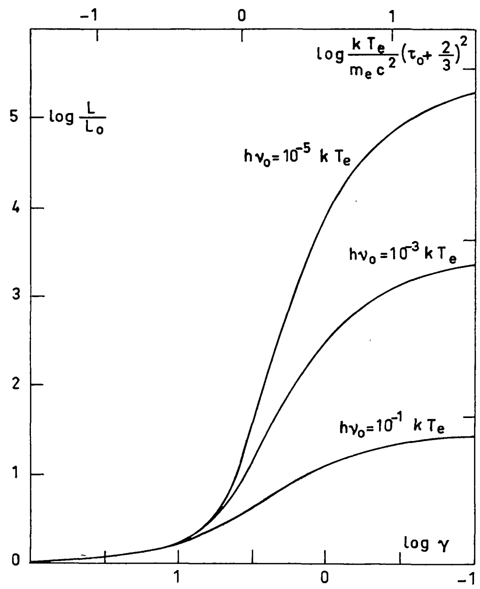


Fig. 7. The ratio of the plasma cloud luminosity L to the luminosity L_0 of sources of low frequency photons as a function of the parameters of the cloud and the energy of low frequency photons. The horizontal dashes in the right part of the drawing correspond to the limit value of $3kT_e/h\nu_0$

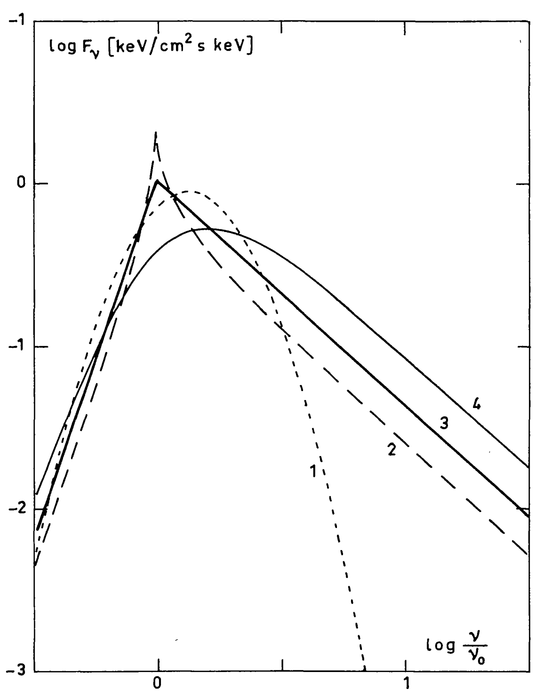


Fig. 8. Influence of comptonization on the monochromatic line profile with $h\nu \ll kT_e$ escaping from the spherical plasma cloud with $\gamma=6$. Only the Doppler change of photons frequency is taken into account. Curve 4 corresponds to the central location of the photon sources, 3 – distribution of the sources according to law (8), 2 – uniform distribution of the sources over the cloud. For comparison curve 1 illustrates the time evolution of the profile of the initially monochromatic line in a homogeneous infinite medium. The profile is given for the moment of time corresponding to $y=(kT_e/m_ec^2)\sigma_T N_e ct=0.1$. All the curves are normalized to the same number of photons

When $x_0=10^{-3}$ we have $\gamma>4.7$. Figure 7 shows the dependence $\frac{L}{L_0}(\gamma)$ for three values of x_0 (see also Shapiro et al., 1976, Pozdnyakov et al., 1976, 1977, 1979a).

When $\gamma\to 0$ the spectrum of the resulting emission is close to the Wien spectrum with $\overline{h\nu}=3kT_e$ and $\frac{L}{L_0}\to 3x_0^{\gamma/3-1}$ according to (23). As the number of photons is constant and $L_0=N_\gamma h\nu_0$, then $L_{\max}=3N_\gamma kT_e$ and $\frac{L_{\max}}{L_0}=\frac{3}{x_0}$. If the sources of low frequency emission had a Planckian spectrum we would obtain $L_0=2.7N_\gamma kT_r$ and $L_{\max}/L_0=T_e/0.9T_r$.

In Table 1 the dependence $d(\gamma)$ is given which yields

$$\frac{L}{L_0}=d(\gamma)x_0^{\alpha-1}$$

when $0\leqq\gamma<4$ and for any $x_0\ll 1$. If $x_0\ll x_1\ll 1$, $0<\alpha<1$ and the flux density $F_\nu(x_1)$ at a given frequency x_1 and distance to the source D are known, we can estimate its luminosity

$$L=4\pi D^2 kT_e d(\alpha)\frac{2\alpha+3}{\alpha(\alpha+3)}F_\nu(x_1)x_1^\alpha\,. \tag{27}$$

V. Evolution of the Radiation Spectrum

1. Solution of the Problem by the Convolution Method

It was described in the Introduction how we can find the spectrum of the radiation escaping from the plasma cloud knowing the distribution $P(t)$ of the photons over the escape time and having the solution of the Kompaneets equation for the infinite medium.

a) Doppler-effect

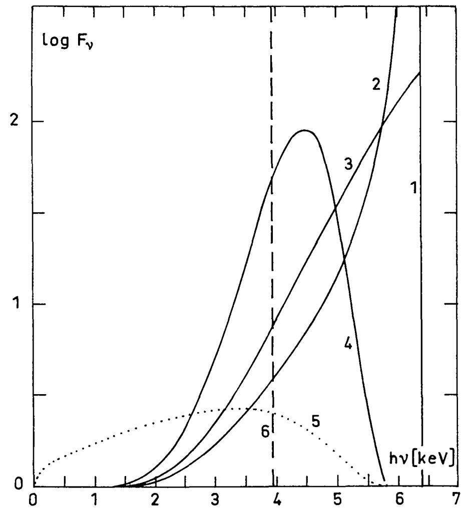
When $kT_e\gg h\nu$ the spectrum evolution is determined by the Doppler-effect. The solution of the Kompaneets equation without the terms n and n^2 which takes into account the recoil effect, is given by formula (4). This solution shows how the line broadens with time and how its center of gravity shifts in the high frequency direction. This solution convolves easily with the asymptotics of the distributions $P(u)$ (Sect. II). The convolution gives the profile of the line, escaping from the plasma cloud if the photon sources in it emit monochromatic line $x_0\cdot\delta(x-x_0)$, $x_0\ll 1$. Let us remind that $y=\frac{kT_e}{m_ec^2}u$. It is clear that the convolution of the solution (4) with the asymptotics $P(u)$ for small $1<u\ll\tau_0^2$ describes the line profile in the vicinity of x_0, the asymptotics with $u\gg\tau_0^2$ determine the spectra in the far wings of the line with $x\ll x_0$ and $x\gg x_0$. It is easy to take or estimate integrals like $\int_0^\infty I_\nu(x,y)P(y)dy$ for four basic types of the $P(u)$ asymptotics (Fig. 2).

When $u\gg\tau_0^2$ the basic asymptotics is $P\sim\exp\left\{-\frac{\pi^2 u}{3(\tau_0+\frac{2}{3})^2}\right\}$. The convolution of this asymptotics with the solution (4) results in power-law spectra like (16). The slope of the far wings of the line is independent of the source distribution throughout the plasma cloud. However the radiation intensity in the wings (their power) is strongly dependent on the source distribution in the cloud.

Comparison of the formulae (5), (5a) and (9) shows that the wings are most powerful in the case where the source of the low frequency photons is located in the centre of the cloud. In the case of the uniform distribution of sources in a spherical cloud the wings are approximately 3 times weaker $\left[\text{the ratio } \frac{P_1(u)}{P_5(u)} = \frac{\pi^2}{3}\right]$. In the case where the sources are distributed according to the formula $\frac{\tau_0}{\pi\tau} \sin \frac{\pi\tau}{\tau_0}$ and where the problem reduces to the solution of the stationary equation we have $\frac{P_7(u)}{P_1(u)} = \frac{1}{2}$. That is the wings are two times weaker than in the case of the central source.

As well as for the stationary equation solution the results can be easily generalized for the case where sources of soft photons have the Planckian spectrum with $T_r \ll T_e$ and for any arbitrary sources spectra with $h\nu < kT_e$, for example for free-free emission of the same plasma cloud. The short wave asymptotics weakly depend on the spectrum of the sources of the low frequency photons. The X-ray luminosity of the cloud $\left(\text{the ratio } \frac{L}{L_0}\right)$ strongly depends on the spatial distribution of these sources. The asymptotic with $u \ll \tau_0^2$ gives the shape of a monochromatic line in the vicinity of x_0. In Fig. 8 the results of the convolution operation are shown. The shape of the curves is strongly dependent on the law of the source distribution over the sphere. With the central source, the line has a broad maximum: practically all the photons having experienced $u \sim \tau_0^2$ scatterings. With a uniform distribution of the sources the boundary effects become important: many photons have experienced a small number of scatterings and practically have not changed their frequency. The solution of the stationary equation has the critical point with $x = x_0$.

For comparison the time dependent solution (4) of Kompaneet's equation for homogeneous infinite medium is shown in this figure. It takes into account only the line broadening due to the Doppler effect, neglecting the effects connected with the difference in time of the photon escape from the cloud. It is seen that the escape time distribution of photons leads to the formation of powerful wings of the line. According to solution (4) these wings are exponentially weak.



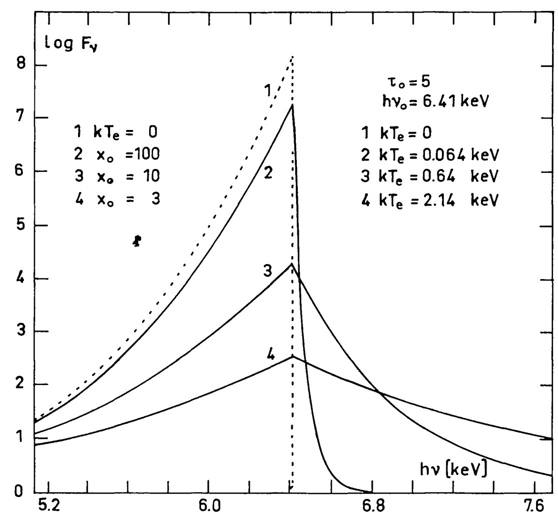


Fig. 9. Effect of comptonization on the profile of the monochromatic line with $h\nu_0 \gg kT_e$ escaping from the plasma cloud. **a** The line profile dependence on photon sources distribution in the cloud with $kT_e = 0$ with the iron K_α line as an example. Curve 1 – the position of the monochromatic line; 2 – the uniform distribution of photon sources; 3 – distribution according to law (8), 4 – the sources are concentrated in the center of the cloud, 5 – the sources are at depth τ_0 in the semi-infinite medium. Vertical line 6 gives the line position in the Kompaneets' equation approximation for the infinite homogeneous medium with $(h\nu_0/m_ec^2)\sigma_T N_e ct = 0.64$. **b** The dependence of the profile on $x_0 = h\nu_0/kT_e$. The sources are distributed according to (8)

b) Recoil Effect. Monochromatic Line Profile Evolution

The scatterings in the cold $kT_e = 0$ plasma lead, due to the recoil effect, to an average increase of the photon wavelength, according to the simple formula

$$\lambda = \lambda_0 + \lambda_c u \tag{28}$$

where λ_0 is the initial wavelength, $\frac{\lambda_c}{2\pi} = \frac{h}{2\pi m_e c}$ is the Compton wavelength and u characterizes the number of scatterings.

To a first approximation one may consider the line to remain monochromatics, shifting only along the frequency axis. The Kompaneets equation has been derived just to this approximation. However in reality after the first scattering event the line broadens from λ_0 to $\lambda_0 + \frac{2h}{m_e c}$ (see the Figures and discussion in the paper by Pozdnyakov et al., 1979a). As we shall demonstrate below the main cause of the profile broadening is the dispersion in the time of photon escape from the plasma cloud and consequently the dispersion in the number of scatterings they underwent. Since the function $P(u)$ is known, it is easy to perform the convolution and obtain the resulting spectrum of the line to a first approximation (neglecting the line broadening in an individual scattering event). It is sufficient for that purpose to use the relation (28) and to replace in $P(u)$ the dimensionless time u by $\frac{\lambda - \lambda_0}{\lambda_c}$ or, which is the same, by

$$u = \frac{m_e c^2}{h\nu} - \frac{m_e c^2}{h\nu_0}.$$

The main parameter of the problem is

$$v = z_0 \tau_0^2 = h\nu_0 \tau_0^2 / m_e c^2.$$

Indeed, the photons are distributed over the escape time according to

$$\frac{dN_\gamma}{du}=P(u).$$

The sources of photons emit the monochromatic line,

$$\frac{dN_\gamma}{dz}=A\delta(z-z_0)$$

and the photons escape from the cloud with the spectrum:

$$F_z=Az\frac{dN_\gamma}{dz}=A\frac{dN_\gamma}{du}\frac{du}{dz}z=\frac{A}{z}P\left(\frac{1}{z}-\frac{1}{z_0}\right). \quad (29)$$

Substituting dimensional variables we find

$$F_\nu=A\frac{m_ec^2}{h\nu}P\left(\frac{m_ec^2}{h\nu}-\frac{m_ec^2}{h\nu_0}\right)=A\frac{\lambda}{\lambda_c}P\left(\frac{\lambda-\lambda_0}{\lambda_c}\right)\left[\frac{\mathrm{keV}}{\mathrm{cm^2\,g\,keV}}\right].$$

The spectrum is normalized to the total number of photons from the cloud detected per unit time

$$\int_0^\infty\frac{F_\nu}{\nu}d\nu=A\left[\frac{\mathrm{photons}}{\mathrm{cm^2\,s\,keV}}\right].$$

Using the formulae for $P(u)$ from part II it is easy to determine the line profile for any distribution of sources over the cloud. Some interesting cases are shown in Fig. 9. In the case of a spherical cloud with the source in the center it is clear from the formulae (5, 6) that the maximum of the flux is at the wavelength $\lambda=\lambda_0+0.3\lambda_c\tau_0^2$, the far wings with $\lambda-\lambda_0\gg0.3\lambda_c\tau_0^2$ and $\lambda-\lambda_0\ll0.3\lambda_c\tau_0^2$ are exponentially weak and the line width is close to $\Delta\lambda_1\sim\lambda_c\tau_0^2$.

If the sources are distributed homogeneously in the cloud the boundary effects are important and many photons escape from the cloud after only few scatterings. So, according to (5a), (7) and (29) the high energy wing of the line is described by

$$F_\nu\sim[\nu(\nu_0-\nu)]^{-1/2} \quad (30)$$

and the low energy wing by

$$F_\nu\sim\frac{1}{\nu}\exp\left\{-\frac{\pi^2m_ec^2}{3\tau_0^2h\nu_0}\frac{\nu_0-\nu}{\nu}\right\}. \quad (31)$$

In the case of a semi-infinite atmosphere with sources being in the zone $0\leqq\tau\leqq\tau_0$ the high-frequency asymptotic of the profile coincides with (30) and the low-frequency one stretches down to zero frequency

$$F_\nu\sim\frac{\nu^{1/2}}{(\nu_0-\nu)^{3/2}}. \quad (32)$$

Only the position of the intensity maximum depends on τ_0.

c) Causes of the Line Broadening

When $kT_e=0$ the line broadens due to: 1. photon escape time dispersion and 2. the line broadening in each scattering event. The first cause as mentioned above leads to the line width of the order of $\Delta\lambda_1\sim\tau_0^2\lambda_c$, the second only to $\Delta\lambda_2\sim\tau_0\lambda_c$. Indeed, the change of the wavelength at each event of scattering $0<\Delta\lambda<2\lambda_c$ may be larger or smaller than the average value λ_c. We are to solve a typical diffusion problem. Therefore the broadening of the line is proportional to the square root of the number of scatterings, i.e.

$$\Delta\lambda_2\sim\lambda_c\sqrt{u}\sim\tau_0\lambda_c, \qquad \Delta\lambda_1\gg\Delta\lambda_2$$

when $\tau_0\gg1$. The Doppler effect becomes the third cause of the line broadening

$$\Delta\lambda=\pm\frac{v}{c}\lambda_0=\pm\left(\frac{2kT_e}{m_ec^2}\right)^{1/2}\lambda_0$$

for scattering on electrons with the finite temperature

$$0\leqq kT_e\leqq h\nu_0.$$

When the number of scatterings is large the line broadens as the square root of the number of scatterings (Pozdnyakov et al., 1979a)

$$\Delta\lambda_3=2\sqrt{\ln2}\left(u\frac{kT_e}{m_ec^2}\right)^{1/2}\approx2\left(\frac{kT_e}{m_ec^2}\right)^{1/2}\tau_0. \quad (33)$$

Doppler broadening may be neglected as compared to $\Delta\lambda_2$ when (Pozdnyakov et al., 1979a)

$$kT_e<\frac{(h\nu_0)^2}{2m_ec^2}. \quad (34)$$

Let us remind that the recoil effect determines the shift of the gravity center of the line when $h\nu_0\gg4kT_e$, i.e. when $x_0>4$.

How correct these estimates are is illustrated by Fig. 9b, where the solutions of the stationary Kompaneets' equation are compared with Eq. (B5) and without the Doppler effect being taken into account Eq. (20a). It is seen that when $x_0=100$, the Doppler and the recoil effects are approximately of the same importance in the line broadening. This is in a good agreement with the estimate (34). At the same time all the diagrams for $x_0>3$ demonstrate the shift of the gravity center of the line to the region of small x.

The spectrum of the lines mostly depends on the distributions of photon sources over the plasma cloud, therefore we do not present detailed plots. The profile strongly depends on the model and, vice versa, observing the profiles of iron X-ray lines it is possible to obtain information about the distribution of photon sources in the plasma cloud.

Due to the existence of $\Delta\lambda_2$ and $\Delta\lambda_3$ the far wings of the line must obey the power law. We took into account only the line broadening due to dispersion in the escape times. Therefore in our approximation the far wings of the line exponentially weak. The real situation is similar to the case of the Voigt profile, where the profile is exponential in the vicinity of the line center and obeys the power law in the far wings. Comparison with Monte-Carlo computations (Pozdnyakov et al., 1979a) shows that our approximation is valid only in the central parts of the line profile.

d) Recoil Effect. Evolution of the Power Spectrum

Any spectrum of the sources of photons can be represented as

$$f(\nu)=\int_0^\infty f(\nu_0)\delta(\nu-\nu_0)d\nu_0. \quad (35)$$

We have given above the solution of the monochromatic line profile evolution problem (29). Using (29) the transformation (35) allows the emission spectrum of the cloud to be obtained, the spectrum of the sources of photons being arbitrary. Substituting the solution (29) for $\delta(\nu-\nu_0)$ in (35) and taking into account that for the monochromatic line $\nu\leqq\nu_0$ is always the case for the approximation considered we find:

$$F_\nu(z)=\frac{1}{z}\int_z^\infty f(z_0)P\left(\frac{1}{z}-\frac{1}{z_0}\right)\frac{dz_0}{z_0}$$
$$=\frac{1}{z}\int_0^{1/z}\frac{1}{1/z-u}f\left(\left(\frac{1}{z}-u\right)^{-1}\right)P(u)du. \quad (36)$$

For the power spectrum $f_\nu(z_0)=Az_0^{-\alpha}$ with

$P(u)=\beta\exp\{-\beta u\}$

the formula (36) is identical to the solution (20) of the stationary equation. In the case of a central source of hard photons with the expressions (5) and (6) substituted for $P(u)$ in (36) it is evident that the high frequency asymptotics of the spectrum $z/\beta\gg 1$ is determined completely by asymptotics $P(u)$ with $u<\tau_0^2$. It is natural because photons with $h\nu_0/m_ec^2>1/\tau_0^2$ can remain the spectrum range considered only if they have experienced much less than the average number of scatterings in the source $u\ll\tau_0^2$. The probability of escape in this case is determined by the formula (6). Substituting it into (36) we obtain

$$F_\nu(z)=\frac{A}{z}\int_0^{1/z}\left(\frac{1}{z}-u\right)^{\alpha-1}u^{-3/2}\exp\left\{-\frac{3}{4}\frac{\tau_0^2}{u}\right\}du\,.$$

When $\alpha>0$ this integral is reduced to the Whittaker function (Gradstein and Ryzhik, 1971, formula 3.471.2), hence it follows that in the high frequency limit

$$F_\nu(z)=\tau_0^2\left(\frac{h\nu}{m_ec^2}\right)^{-2\alpha+1/2}\exp\left\{-\frac{3}{4}\tau_0^2\frac{h\nu}{m_ec^2}\right\}. \qquad (37)$$

In the case of homogeneous distribution of the sources in the cloud the probability of a photon undergoing only few scatterings as compared with the average is determined by (7). We have

$$F_\nu(z)\cong\frac{1}{\sqrt{z}}\int_z^\infty\frac{z_0^{-1/2-\alpha}}{(z-z_0)^{1/2}}dz_0\sim z^{-\alpha-1/2}\,. \qquad (38)$$

Thus, the shape of the resulting spectrum depends crucially on the distribution character of hard photon sources in the cloud (Fig. 10). In the case of power law spectrum of the photon sources $F_\nu(z)=Az^{-\alpha}$, $\alpha>0$ the spectrum of escaping photons has a break at $h\nu\geqq m_ec^2/\tau_0^2$

a) $\alpha\to\alpha+\frac{1}{2}$ in the case of a homogeneous distribution of the sources;

b) $\alpha\to\alpha+1$ when the density of sources changes according to the law $\frac{\pi\tau_0}{\tau}\sin\frac{\pi\tau}{\tau_0}$ (see the formula (21));

c) the spectrum cuts off exponentially if the source is situated in the center of the cloud.

e) Solution of Comptonization Problem for Arbitrary Distribution of Sources in a Finite Plasma Cloud

In Sect. III.4 the solution of the stationary Kompaneets equation was found for the spectrum of emission from an isothermal cloud with a uniform density distribution of the plasma, the photon sources being distributed according to the law $\frac{\pi\tau_0}{\tau}\sin\frac{\pi\tau}{\tau_0}$. Only in this case $P(u)=\beta\exp(-\beta u)$. The solution is true for any relation between $h\nu$ and kT_e and for any spectrum of the photon sources.

There is a challenge to obtain the full solution of the problem by an arbitrary plasma density and sources distribution over the volume of an isothermal cloud of an arbitrary geometry.

This can be done quite easily if one manages to separate the spatial and temporal coordinates in the diffusion equation and find the law of photon escape from the cloud $P(t)$ Function $P(u)$ can be expanded in a series with the terms of

$$P(u)=\sum_{k=1}^\infty c_k\beta_k\exp\{-\beta_ku\}\,.$$

The convolution of $P(u)$ with the solution of Kompaneets' equation $J_\nu(x,u)$ can be represented as a combination of solutions

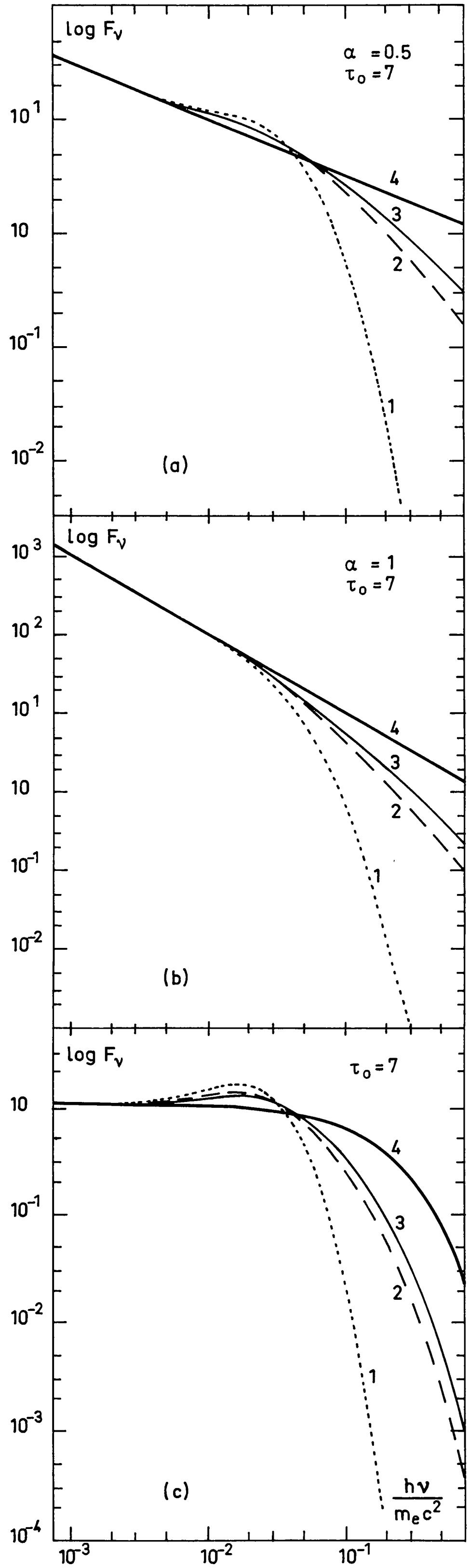


Fig. 10. Hard X-ray $h\nu\gg kT_e$ radiation spectrum evolution due to comptonization. Only the recoil effect is taken into account. The sources of photons in Fig. 10a and b have a power-law spectra. The photon sources in Fig. 10c have the spectrum of bremsstrahlung of an optically thin layer with $T_r\gg T_e$. Bold curves: the emission spectra of the photon sources; solid: the spectrum of escaping radiation from the cloud with uniformly distributed photon sources; dashed: the same for the distribution according to law (8); dotted: the same for the central location of the sources

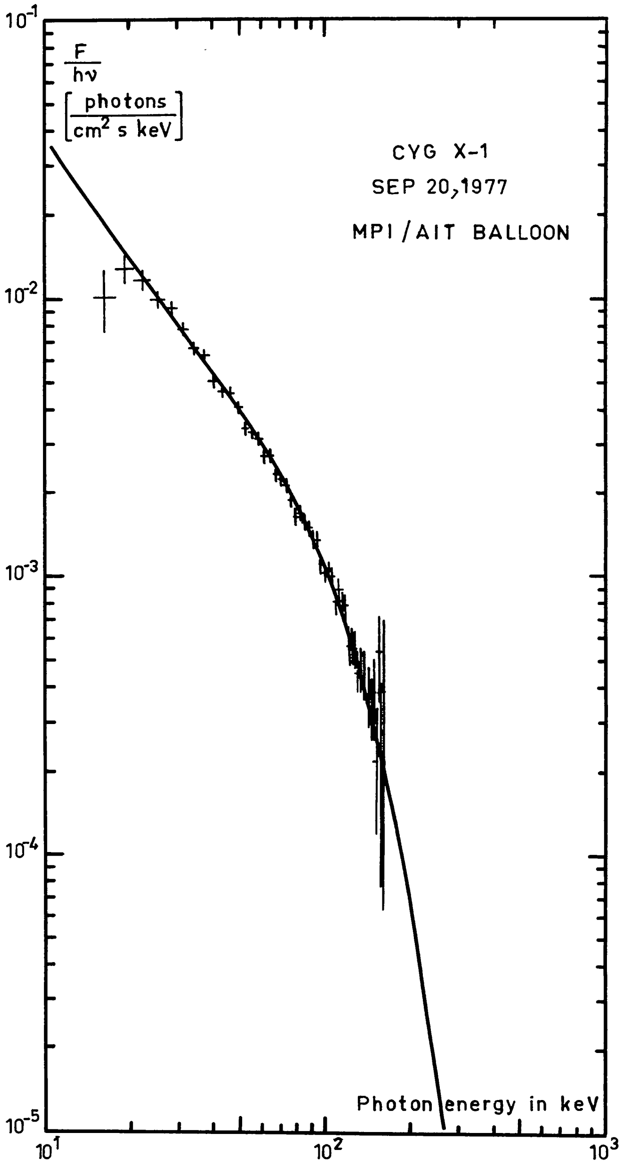


Fig. 11. Comparison of the observed Cyg X-1 radiation spectrum (Voges et al., 1979) with the spectrum resulting from comptonization of low frequency photons in the plasma cloud with $\tau_0 = 5$ and $kT_e = 27$ keV

of the stationary Eq. (13) related to different parameters β_k

$$F_\nu(x) = \int_0^\infty J_\nu(x,u) \sum_k c_k \beta_k \exp\{-\beta_k u\} du$$

$$= \sum_k c_k I_\nu^k(x).$$

Here

$$I_\nu^k(x) = \beta_k \int_0^\infty J_\nu(x,u) \exp\{-\beta_k u\} du$$

is the solution of the stationary Eq. (13) with $\beta = \beta_k$. This method enables us to find the emission spectrum and the luminosity L of the plasma cloud, with the given spectrum and luminosity L_0 of photon sources.

As an example of this application let us find the luminosity of a spherical plasma cloud with

$$y = \frac{kT_e}{m_e c^2} \tau_0^2 \ll 1$$

and a central source of photons with $h\nu \ll kT_e$.

In the Appendix it is shown (A8) that if there is a central source, then:

$$\beta_n = \lambda_n^2/3$$

where λ_n meets the transcendental equation $\mathrm{tg}\, \lambda_n \tau_0 = -\frac{2}{3}\lambda_n$.

The solution of this equation with $\tau_0 \gg 1$ is

$$\lambda_n \tau_0 = n\pi - \sum_{k=1}^{n} \alpha_k$$

where

$$\alpha_k = \frac{2\pi/3\tau_0}{1 + 2/3\tau_0 + 4\lambda_{k-1}^2/9}, \quad \lambda_0 = 0, \quad k = 1, 2, 3 \ldots . \tag{39}$$

The normalization condition for $P(u)$ gives an equation

$$3 \sum_{n=1}^{\infty} \frac{\sin \lambda_n \tau_0}{\lambda_n} = 1.$$

The formula for $P(u)$ allows the cloud luminosity to be derived with Doppler effect being the main factor of photon energy change. In this case for every β_n, we have

$$\frac{L_n}{L_0} = \frac{\gamma_n}{\gamma_n - 4}, \quad \gamma_n = \beta_n \frac{m_e c^2}{kT_e}$$

so

$$\frac{L}{L_0} = 3 \sum_{n=1}^{\infty} \frac{\gamma_n}{\gamma_n - 4} \frac{\sin \lambda_n \tau_0}{\lambda_n}.$$

When $\tau_0 \gg 1$ Eq. (39) gives $\gamma_n \approx n^2 \gamma_1$. Note that

$$\gamma_1 = \pi^2 m_e c^2 / 3(\tau_0 + \tfrac{2}{3})^2 kT_e$$

coincides with the definition of γ (14). When $\gamma_1 > 4$, $\gamma_2 \gg 4$, $\gamma_3 \gg 4$, and so on, i.e. $\frac{\gamma_n}{\gamma_n - 4} \approx 1$ when $n > 2$ the expression for L/L_0 transforms into

$$\frac{L}{L_0} = \frac{3\gamma_1}{\gamma_1 - 4} \frac{\sin \lambda_1 \tau_0}{\lambda_1} + 3 \sum_{n=2}^{\infty} \frac{\sin \lambda_n \tau_0}{\lambda_n}$$

$$= 3\left(\frac{\gamma_1}{\gamma_1 - 4} - 1\right) \frac{\sin \lambda_1 \tau_0}{\lambda_1} + 3 \sum_{n=1}^{\infty} \frac{\sin \lambda_n \tau_0}{\lambda_n}$$

$$= 1 + \frac{8}{\gamma - 4}\left(1 - \frac{\alpha_1^2}{2}\right) \approx \frac{\gamma + 4}{\gamma - 4}$$

here

$$\alpha_1 = \frac{2\pi}{3} / (\tau_0 + \tfrac{2}{3}).$$

It is easy to show by the same method that the luminosity of a spherical cloud uniformly filled with plasma and sources of photons is given as

$$\frac{L}{L_0} \approx 1 + \frac{24}{\pi^2(\gamma_1 - 4)\left(1 + \frac{4\pi^2}{9\tau_0^2}\right)} \approx \frac{\gamma_1 - 1.6}{\gamma_1 - 4}.$$

In this case $P(u)$ is respectively:

$$P(u)=\frac{6}{\pi^2}\beta_1\sum_{n=1}^{\infty}\frac{\exp\{-\beta_n u\}}{1+\frac{4}{9}\lambda_n^2}.$$

With the central source of photons the energy loss is $\frac{\gamma_1+4}{\gamma_1-1.6}$ times that in the case of uniform distribution of the sources. For $\gamma_1=6$ this ratio is equal to 2.2.

V. Astrophysical Applications

1. Bursters

The X-ray bursts observed have a steep front (typical rise time is less than 1 s). Then an exponential decrease of the intensity follows, the typical decrease time being of the order 6–8 s (Gursky et al., 1976; Lewin and Joss, 1977). Canizares (1977) suggested that the light curve of the burster reflects the law of the photon distribution over their escape time t from the cloud. He calculated $P(t)$ for small $\tau_0<10$ using the Monte-Carlo method. The solution obtained in Sect. II (Fig. 1) gives the function $P(t)$ shape similar to that observed from many bursters.

Let us consider a spherical cloud of completely ionized plasma of radius R having optical depth τ_0 with respect to Thomson scattering.

The source of photons is in the center. Let us consider an instantaneous flare of emission. Function $P(t)$ (Fig. 1) allows us to determine some parameters of the sources. According to (5) the typical time of the intensity decrease $\exp\{-t/t_0\}$ corresponds to

$$t_0=3\tau_0^2/\pi^2\sigma_T N_e C=\frac{3}{\pi^2}\frac{R\tau_0}{c},$$

i.e. with R given, it is easy to find τ_0 and the density of the plasma. Similar estimates are valid for the hot layer in the semi-infinite plane atmosphere with photon sources situated at optical depth τ_0. The more deep layer must have albedo $A=0$.

The burster phenomenon is likely related to accretion onto neutron stars. Unstable thermonuclear burning of helium and carbon in the matter fallen onto the surface of a neutron star may result in bursts (see the review and the references in the papers by Lamb and Lamb, 1977; Joss, 1977). The layer where the burning occurs must have a temperature of about 10^9 K. The typical depth of a layer with such a temperature (the height of the homogeneous atmosphere) is close to

$$H=2R^2kT/GMm_p=4R^2kT/Rgm_pc^2$$
$$=15\left(\frac{T}{10^9\,\mathrm{K}}\right)\left(\frac{R}{10\,\mathrm{km}}\right)\left(\frac{M_\odot}{M}\right)\mathrm{m}.$$

Suppose that the energy release lasts less than 1 s, then for the typical albedo of the low atmospheric layers being close to 1, the light curve must be similar to that in Fig. 1.

The decay time in this case is $t_0=\frac{3}{\pi^2}\frac{H}{c}\tau_0$. Assuming $H=15$ m and $t_0=6$ s we find $\tau_0=4\,10^8$ and $\varrho=m_p\tau_0/\sigma_T H=6\,10^5\,\mathrm{g\,cm^{-3}}$ densities of this order of magnitude are expected to be in a zone of unstable burning. When the temperature and the density have such values, the electrons are not yet degenerate and the scattering gives the main contribution to the opacity, i.e. determines the escape time and the type of function $P(t)$. When $T<5\,10^8$ K a considerable contribution to the opacity is from free-free processes. It is possible to take into account degeneration of electrons: nevertheless it is clear that the shape of the curve will be similar to that in Fig. 1. Important is only the fact that a small fraction of the energy released enters the zone with $\tau\gg\tau_0$, heats it and leads to the production of low frequency photons. Then these photons are comptonized in hot layer.

2. Temperature of the Plasma in Cyg X–1

There is a number of models explaining the origin of the hard X-ray flux from Cyg X–1. Shapiro et al. (1976) consider it as a manifestation of a two-temperature accretion disk. Shakura and Sunyaev (1976) consider it to be the emission from hot zones, forming due to thermal and secular instabilities in the disk. Thorne and Price (1976), Bisnovatyi-Kogan and Blinnikov (1977) believe it to be the emission from the hot corona above the disk. The authors of all these models are unanimous in the opinion that the spectrum is formed due to comptonization of low frequency photons on hot electrons (Eardley et al., 1978).

The formulae obtained in Sect. III.4 allow the temperature and the scattering optical depth of the hot plasma in the region of the main energy release in Cyg X–1 to be found.

The spectrum of the emission escaping the cloud is described by (23) and in the hard X-ray region $\nu\gg\bar{\nu}_0$ depends only on one parameter

$$\gamma=\pi^2m_ec^2/3(\tau_0+\tfrac{2}{3})^2kT_e.$$

Figure 11 shows experimental data on Cyg X–1, obtained by Voges et al. (1979), see also Sunyaev and Trümper, 1979.

These are compared with the calculations made using the formula (23a) with $\gamma=2.0$ and $kT_e=27$ keV (the shape of the curve $F_\nu(x)$ is determined completely by γ): if we shift the curve along the frequency axis we get it matched with the experimental points, hence the correspondence between x and $h\nu$ and the value of kT_e are obtained. Although the emission spectrum extends up to 150 keV, the radiation may be emitted by a plasma cloud with the temperature $kT_e=27$ keV and $\tau_0=5$, $\alpha=0.56$.

3. X-ray Radiation of the Nuclei of Galaxies and Quasars

If X-ray emission spectrum of galactic nuclei and quasars is actually formed due to comptonization of the low-frequency radiation scattered on hot electrons, the method described in the previous section enables us to find the temperature and the optical depth of the plasma clouds where the spectrum forms.

Using the formulae in Sect. III and Table 1 it is easy to find the $F_\nu(x)$ for a wide range of γ-values and spectral indices. The relation between the low frequency source luminosity and the X-ray luminosity obtained in Sect. IIIg can be useful for constructing detailed models.

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Appendix A

Photon Distribution over the Time of Escape from a Spherical Plasma Cloud

A spherical plasma cloud with distributed photon sources is considered. The Thomson scattering is the main mechanism of photon interaction with the plasma. The equation of photon diffusion in the cloud is

$$\frac{\partial J}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D(r)\frac{\partial J}{\partial r}\right) \tag{A.1}$$

where the diffusion coefficient is $D=\dfrac{C}{2\sigma_T N_e(r)}$, the average intensity of the emission is $J(r,t)$. In the homogeneous plasma

$$\frac{\partial J}{\partial u}=\tfrac{1}{3}\frac{1}{\tau^2}\frac{\partial}{\partial \tau}\left(\tau^2\frac{\partial J}{\partial \tau}\right). \tag{A.2}$$

Here $u=c\sigma_T N_e t$ is the dimensionless time τ, is the optical distance to the centre of the cloud.

The boundary condition

$$\left.\frac{\partial J}{\partial \tau}+\tfrac{3}{2}J\right|_{\tau=\tau_0}=0 \tag{A.3}$$

shows that there is no photon flux from the outside onto the boundary surface of the cloud. $\tau_0=\sigma_T N_e R$ characterizes the radius of the spherical cloud.

The initial condition

$$J|_{u=0}=f(\tau) \tag{A.4}$$

shows that at the time equal to zero the sources distributed over the cloud according to the law $f(\tau)$ were turned on. The solution of the problem with general initial conditions (A.4) can be obtained as a linear superposition of the solutions with a condition

$$J|_{u=0}=\frac{\delta(\tau-a)}{4\pi a^2}. \tag{A.5}$$

Condition (A.5) shows that at the time $t=0$ there was a source of photons turned on a sphere with the radius and with the power 4π. In the limit $a\to 0$ it is possible to obtain the solution for a point source in the center of the sphere.

The solution of the mixed boundary problem (A.2, A.3, A.5) can be obtained by separation of variables method in the form (Petrovsky, 1953)

$$\begin{aligned} J(\tau,u)&=\sum_{n=1}^{\infty}\beta_n\frac{\sin\lambda_n\tau}{\tau}\exp\left\{-\frac{\lambda_n^2 u}{3}\right\},\\ \beta_n&=\frac{1}{4\pi a I_n}\sin\lambda_n a,\\ I_n&=\tfrac{1}{2}\left(\tau_0+\frac{(\frac{3}{2}\tau_0-1)\tau_0}{(\frac{3}{2}\tau_0-1)^2+(\lambda_n\tau_0)^2}\right),\\ \mathrm{tg}\,\lambda_n\tau_0&=\frac{\lambda_n\tau_0}{1-\frac{3}{2}\tau_0}. \end{aligned} \tag{A.6}$$

Below we shall discuss the case of the great optical depth $\tau_0\gg 1$ and $u>1$. For $\tau_0\gg 1$

$$J(\tau_0,u)=\frac{1}{2\pi a\tau_0^2}\sum_{n=1}^{\infty}\sin\lambda_n a\sin\lambda_n\tau_0\exp\left\{-\frac{\lambda_n^2 u}{3}\right\}, \tag{A.7}$$

$$\mathrm{tg}\,\lambda_n\tau_0=-\tfrac{2}{3}\lambda_n. \tag{A.8}$$

For $a=0$

$$J(\tau_0,u)=\frac{1}{2\pi\tau_0^2}\sum_{n=1}^{\infty}\lambda_n\sin\lambda_n\tau_0\exp\left\{-\frac{\lambda_n^2 u}{3}\right\}. \tag{A.9}$$

The series obtained converge fast when $\dfrac{\lambda_1 u}{3}\gg 1$ or $u\gg\dfrac{3}{\pi^2}\tau_0^2$ and slowly when $u\ll\tau_0^2$. When $u\gg\dfrac{3}{\pi^2}\tau_0^2$ the series (A.7) and (A.9) are described accurately enough by the first term. When $a\ll 1$ we have

$$J(\tau_0,u)\cong\frac{\pi}{3\tau_0^4}\exp\left\{-\frac{\pi^2 u}{3\tau_0^2}\right\}. \tag{A.10}$$

To obtain the asymptotic behaviour with $u\ll\tau_0^2$ let us construct the solution of the problem (A.2, A.3, A.5) using the reflection method. Let us write down the final expression (Sunyaev and Titarchuk, 1978)

$$\begin{aligned} J_a(\tau_0,u)&=\frac{1}{4\pi\tau_0 a}\left(\frac{3}{\pi u}\right)^{1/2}\\ &\cdot\left[\exp\left(-\frac{3(\tau_0-a)^2}{4u}\right)-\exp\left(-\frac{3(\tau_0+a)^2}{4u}\right)\right.\\ &-h\int_0^{\infty}\left(\exp\left(-\frac{3(\tau_0+a+\eta)^2}{4u}\right)\right.\\ &\left.\left.-\exp\left(-\frac{3(\tau_0-a+\eta)^2}{4u}\right)\right)\exp(-h\eta)d\eta\right]+R(\tau_0,a,u). \end{aligned} \tag{A.11}$$

Here the terms in brackets determine the asymptotics with $u\ll\tau_0^2$. $R(\tau_0,a,u)$ is the remainder of the series. When $a=0$ we have

$$\begin{aligned} J_0(\tau_0,u)&=\frac{h}{4\pi\tau_0}\left(\exp\left(-\frac{3\tau_0^2}{4u}\right)\right.\\ &\left.-h\int_0^{\infty}\exp\left(-\frac{3(\tau_0+\eta)^2}{4u}\right)\exp(-h\eta)d\eta\right)+R(\tau_0,0,u). \end{aligned}$$

Hence it is not difficult to obtain that, with $1\ll u\ll\tau_0^2$,

$$J_0(\tau_0,u)\simeq\left(\tfrac{1}{2}\left(\frac{3}{\pi u}\right)^{1/2}\right)^3\exp\left\{-\frac{3\tau_0^2}{4u}\right\}.$$

From the astrophysical point of view, it would be interesting to solve the problem of the time of photon escape from a spherical plasma cloud, uniformly filled with sources of photons.

The mathematic problem is formulated as (A.2), the initial condition $J(\tau,0)=1$ being added. The solution is represented by a series

$$J(\tau_0,u)=\frac{4}{3\tau_0}\sum_{n=1}^{\infty}\frac{\exp\{-\lambda_n^2 u/3\}}{1+4\lambda_n^2/9}.$$

Hence it is easy to obtain the asymptotics with $1\ll u\ll\tau_0^2$

$$J(\tau_0,u)\sim\frac{2}{\sqrt{3\pi u}}\left(1-\frac{2}{\tau_0}\left(\frac{u\pi}{3}\right)^{1/2}\right)$$

and $u\gg\tau_0^2$

$$J(\tau_0,u)\cong\frac{4}{3\tau_0}\exp\left\{-\frac{\pi^2 u}{3(\tau_0+\frac{2}{3})^2}\right\}.$$

To find the escape time distribution function of the escape time distribution $P(u)$ of photons we have to normalize the function obtained $J(\tau_0,u)$ so that $\int_0^{\infty}P(u)du=1$,

$$P(u)=\frac{1}{\int_0^{\infty}J(\tau_0,u)du}J(\tau_0,u).$$

The normalization is made in the following way. Let $J(\tau, u)$ be the solution of the following combined problem

$$3\frac{\partial J}{\partial u} = \Delta J, \quad 0 \leqq \tau \leqq \tau_0, \quad u > 0, \quad \Delta = \frac{1}{\tau^2}\frac{\partial}{\partial \tau}\tau^2\frac{\partial}{\partial \tau}$$

$$\frac{\partial J}{\partial \tau} + \tfrac{3}{2}J\Big|_{\tau=\tau_0} = 0, \quad u > 0,$$

$$J|_{u=0} = f(\tau) 0 \leqq \tau \leqq \tau_0.$$

Let us integrate the first equation over a sphere of the radius τ_0.

$$\int_{D_{\tau_0}} \Delta J dV = 3 \int_{D_{\tau_0}} \frac{\partial J}{\partial U} dV.$$

Hence using the Gauss-Ostrogradsky formula we obtain

$$\int_{S_{\tau_0}} \frac{\partial J}{\partial \tau} dS_{\tau_0} = 3 \int_{D_{\tau_0}} \frac{\partial J}{\partial u} dV$$

or

$$4\pi\tau_0^2 \frac{\partial J}{\partial \tau} = 3 \int_{D_{\tau_0}} \frac{\partial J}{\partial u} dV,$$

$$-4\pi\tau_0^2 \tfrac{3}{2} J(\tau_0, u) = 3 \int_{D_{\tau_0}} \frac{\partial J}{\partial u} dV.$$

Integrating this relation over u we obtain

$$I = \int_0^\infty J(\tau_0, u) du = \frac{1}{2\pi\tau_0^2} \int_{D_{\tau_0}} f(\tau) dV.$$

For instance, for $f_0(\tau) = \frac{\delta(\tau)}{4\pi\tau^2}$ and $f_v(\tau) = 1$ we have, respectively $I_0 = 1/2\pi\tau_0^2$, $I_v = \frac{2\tau_0}{3}$. Taking into account the normalization obtained, let us write down the formulae for $P(u)$ for two interesting cases:

1. the central source – $P_0(u)$
2. the sphere uniformly filled with sources – $P_v(u)$:

$$P_0(u) = \sum_{n=1}^{\infty} \lambda_n \sin \lambda_n \tau_0 \exp\left\{-\frac{\lambda_n^2 u}{3}\right\};$$

$$P_v(u) = \frac{2}{\tau_0^2} \sum_{n=1}^{\infty} \frac{\exp\{-\lambda_n^2 u/3\}}{1 + 4\lambda_n^2/9}.$$

Comparison of $P(u)$ obtained and functions $P(u)$ for the radius dependent absorption coefficients $\alpha_n(r) \sim r^{-n}$, $\alpha(r) \sim \exp\left\{-\frac{r - r_0}{R}\right\}$ indicates that the difference of their asymptotic with $u \gg \tau_0^2$ and $u \ll \tau_0^2$ is negligible. These asymptotics have in common an exponential cut off by $u \gg \tau_0^2$ with approximately the same exponential factor depending on the dimensions of the cloud, and an exponential rise by $u \ll \tau_0^2$ for the source inside the cloud.

Appendix B

Solution of the Stationary Kompaneets Equation

The problem of a spectrum forming in a spherical cloud with the distribution of photon, corresponding to some of the eigenfunctions in the diffusion problem reduces to finding the solution of the stationary Kompaneets Eq. (13) finite on the semiaxis $x > 0$. To find a complete solution of the boundary problem one should only find a solution generated by the monochromatic line at a frequency x_0.

In this case $f(x)$ is equal to $\frac{\gamma\delta(x - x_0)}{x^3}$.

The differential operator of Eq. (13) is to be reduced to the self-conjugate form. For this purpose we multiply Eq. (13) by the integrating factor $M(x) = \exp\{x\}$

$$\frac{d}{dx}\left(x^4 e^x \frac{dn}{dx}\right) - (\gamma x^2 - 4x^3) e^x n = -\frac{\gamma\delta(x - x_0)}{x_0} e^x \quad \text{(B.1)}$$

The Green function for the self-conjugate problem is (see Petrovsky, 1953)

$$G(x, x_0) = \begin{cases} \frac{1}{\Delta(0)p(0)} X_2(x_0) X_1(x), & 0 \leqq x \leqq x_0 \\ \frac{1}{\Delta(0)p(0)} X_1(x_0) X_2(x), & x_0 \leqq x < \infty. \end{cases} \quad \text{(B.2)}$$

$X_1(x)$ and $X_2(x)$ fit Eq. (13) with $f(x) = 0$ and the condition of the finite solution with $x \to 0$ and $x \to \infty$ respectively. Here $\Delta(0)$ is the Wronski determinant for $X_1(x)$ and $X_2(x)$ calculated in the point $x = 0$ and $p(0)$ is the value of $p(x) = x^4 e^x$ for $x = 0$. The solution of homogeneous Eq. (13) is represented by the Whitteker function (Whitteker and Watson, 1963)

$$X_1(x) = x^{-2} \exp(-x/2) M_{2,(9/4+\gamma)^{1/2}}(x),$$

$$X_2(x) = x^{-2} \exp(-x/2) W_{2,(9/4+\gamma)^{1/2}}(x).$$

Here $M_{2,(9/4+\gamma)^{1/2}}$ and $W_{2,(9/4+\gamma)^{1/2}}$ are Whitteker functions approaching zero with $x \to 0$, ∞ correspondingly.

$$M_{2,(9/4+\gamma)^{1/2}}(x) = \exp(-x/2) x^{1/2+(9/4+\gamma)^{1/2}}$$
$$\varphi((9/4+\gamma)^{1/2} - \tfrac{3}{2}, 1 + (9+4\gamma)^{1/2}, x), \quad \text{(B.3)}$$

$W_{2,(9/4+\gamma)^{1/2}}(x)$ is given by the formula (24). $\varphi(\alpha, \beta, x)$ is a degenerate hypergeometric function. Using representations (B.3) and (24) we obtain

$$\frac{1}{\Delta(0)P(0)} = \frac{\Gamma(\alpha)}{\Gamma(2\alpha + 4)}. \quad \text{(B.4)}$$

Let us write down the final formula for the spectral density of the emission escaping the cloud and being a response to the monochromatic line

$$F_v(x) = x^3 n(x, x_0) = \frac{(\alpha+3)\Gamma(\alpha+1)}{\Gamma(2\alpha+4)} \frac{\exp(x_0/2)}{x_0^3}$$
$$\cdot \varphi(\alpha, 2\alpha+4, x) x^{3+\alpha} \exp(-x)$$
$$\cdot W_{2,(9/4+\gamma)^{1/2}}(x_0) \quad \text{when} \quad 0 \leqq x \leqq x_0, \quad \text{(B.5)}$$
$$F_v(x) = \frac{(\alpha+3)\Gamma(\alpha+1)}{\Gamma(2\alpha+4)} \varphi(\alpha, 2\alpha+4, x_0)$$
$$\cdot x_0^{\alpha-1} x \exp(-x/2)$$
$$\cdot W_{2,(9/4+\gamma)^{1/2}}(x) \quad \text{when} \quad x_0 \leqq x < \infty.$$

Here $n(x, x_0)$ is the finite solution of Eq. (B.1), $\alpha = (9/4+\gamma)^{1/2} - \frac{3}{2}$.

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